INDIVISIBILITY AND UNIFORM COMPUTATIONAL STRENGTH

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Abstract. A countable structure is indivisible if for every coloring with finite range there is a monochromatic isomorphic subcopy of the structure. Each indivisible structure naturally corresponds to an indivisibility problem which outputs such a subcopy given a presentation and coloring. We investigate the Weihrauch complexity of the indivisibility problems for two structures: the rational numbers $\mathbb{Q}$ as a linear order, and the equivalence relation $\mathscr{E}$ with countably many equivalence classes each having countably many members. We separate the Weihrauch degrees of both corresponding indivisibility problems from several benchmarks, showing in particular that the indivisibility problem for $\mathbb{Q}$ cannot solve the problem of finding a monochromatic rational interval given a coloring for which there is one; and that the Weihrauch degree of the indivisibility problem for $\mathscr{E}$ is strictly between those of RT^2 and SRT^2, two widely studied variants of Ramsey's theorem for pairs whose reverse-mathematical separation was open until recently.

1. Introduction

Reverse mathematics is a branch of logic which seeks to classify theorems based on their logical strength. This is often done by working over a relatively weak base system for mathematics, such as a fragment of second-order arithmetic, and then looking at the truth of various implications in models of that fragment. It turns out that one can frequently obtain more fine-grained distinctions between theorems by comparing their intrinsic computational or combinatorial power, rather than being limited to considerations of provability only. For theorems expressible as Π^1_2 statements of second-order arithmetic, this can be achieved by identifying a theorem with an instance-solution pair, where instances are all sets X to which the hypothesis of the theorem apply, and solutions to X are all sets Y witnessing the truth of the conclusion of the theorem for X. One then introduces various reducibilities to

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compare, among different instance-solution pairs, the difficulty of finding a solution from a given instance.

Instance-solution pairs can be formalized as problems:

Definition 1.1. A *problem* is a partial multivalued function on Baire space, $P\colon\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Any $x \in \operatorname{dom} P$ is called an *instance* of P, and any $y \in P(x)$ is a *solution* to x with respect to P.

In the present work, we focus on problems arising from indivisible structures, and use mainly Weihrauch reducibility to gauge their uniform computational content. We assume familiarity with computability theory and the basic terminology of computable structure theory, as in the introductory sections of [Mon21]. All our structures are countable and we will take them to have domain $\mathbb{N}$ when convenient.

Definition 1.2. A structure $\mathcal{S}$ is *indivisible* if for any presentation $\mathcal{A}$ of $\mathcal{S}$ and any coloring of $A = \operatorname{dom} \mathcal{A}$ with finitely many colors, there is a monochromatic subset $B \subseteq A$ such that $\mathcal{B} \simeq \mathcal{A}$, where $\mathcal{B}$ is the substructure induced by B. $\mathcal{S}$ is *computably indivisible* if in addition such a set B can always be computed from $\mathcal{A}$ and c.

Any single-element structure is indivisible, but otherwise an indivisible structure must be infinite, and we will take all structures to be infinite unless otherwise specified. Indivisibility belongs properly to Ramsey theory, the study of combinatorial structures which cannot be "too disordered" in that any large subset must contain a highly organized substructure. This field goes back to 1930, when Ramsey's theorem first appeared, and principles arising in Ramsey theory have long been objects of interest in reverse mathematics.

Definition 1.3. If $\mathcal{S}$ is an indivisible structure, then $\mathsf{Ind}\,\mathcal{S}$ is the *indivisibility problem* associated to $\mathcal{S}$, for which

- Instances are triples $\langle \mathcal{A}, c, k\rangle$ where $\mathcal{A}$ is a presentation of $\mathcal{S}$ (identified with its atomic diagram) and c is a k-coloring of A, and
- Solutions to $\langle \mathcal{A}, c, k\rangle$ are (characteristic functions of) subsets B of A which are monochromatic for c and such that $\mathcal{B} \simeq \mathcal{A}$.

Here $\mathcal{A}$ is allowed to be an arbitrary presentation of $\mathcal{S}$, not necessarily one having domain $A = \mathbb{N}$. We still treat c as a coloring of all of $\mathbb{N}$ but disregard any color assigned to a point outside A. By convention, $i \in A$ iff the formula $x_i = x_i$ is true in the atomic diagram of $\mathcal{A}$, and if it is false then all atomic formulas involving x_i should also be set to false. We will also use the following variations on $\mathsf{Ind}\,\mathcal{S}$:

- For each fixed k, let $\mathsf{Ind}\,\mathcal{S}_k$ have domain $\{\, \langle \mathcal{A}, c\rangle : \langle \mathcal{A}, c, k\rangle \in \operatorname{dom}(\mathsf{Ind}\,\mathcal{S}) \,\}$ and let $\mathsf{Ind}\,\mathcal{S}_k(\langle \mathcal{A}, c\rangle) = \mathsf{Ind}\,\mathcal{S}(\langle \mathcal{A}, c, k\rangle)$. That is, $\mathsf{Ind}\,\mathcal{S}_k$ is the restriction of $\mathsf{Ind}\,\mathcal{S}$ to k-colorings, so that $\mathsf{Ind}\,\mathcal{S}$ is exactly the problem $\bigsqcup_{k\in\mathbb{N}} \mathsf{Ind}\,\mathcal{S}_k$, where $\sqcup$ is defined in the next section.
- Let $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ be the same problem as $\mathsf{Ind}\,\mathcal{S}$ except without any given information on the number of colors used. In other words, $\operatorname{dom}(\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}) = \bigcup_{k\in\mathbb{N}} \operatorname{dom}(\mathsf{Ind}\,\mathcal{S}_k)$ and $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}(x) = \mathsf{Ind}\,\mathcal{S}_k(x)$ if $x \in \operatorname{dom}(\mathsf{Ind}\,\mathcal{S}_k)$.

One can also define the *strong indivisibility problem* where a solution additionally includes an explicit isomorphism between $\mathcal{B}$ and $\mathcal{A}$. We do not study this notion here; both it and $\mathsf{Ind}\,\mathcal{S}$, or more specifically what we call $\mathsf{Ind}\,\mathcal{S}_2$, are discussed in unpublished work of Ackerman, Freer, Reimann, and Westrick [AFRW19].

This paper focuses on two indivisible structures in particular. First, the set of rational numbers $\mathbb{Q}$ is viewed as, up to order isomorphism, the unique countable dense linear order with no greatest or least element. $\mathsf{Ind}\,\mathbb{Q}$ is studied in Section 4. The other structure is the so-called countable equivalence relation:

Definition 1.4. The *countable equivalence relation* $\mathscr{E}$ is, up to isomorphism, the structure in the language $\{R\}$ of a single binary relation such that R is an equivalence relation, and such that E is divided into countably many equivalence classes each having countably many members.

To see $\mathscr{E}$ is indivisible, suppose its elements are colored red and blue (it suffices to show for two colors). Then either there are infinitely many equivalence classes each containing infinitely many red points, or there are cofinitely many equivalence classes each containing cofinitely many blue points. Either way one gets a monochromatic subcopy of $\mathscr{E}$. $\mathsf{Ind}\,\mathscr{E}$ is studied in Section 5. Our main results separate $\mathsf{Ind}\,\mathbb{Q}$ and $\mathsf{Ind}\,\mathscr{E}$ from various benchmark problems. In the case of $\mathbb{Q}$, we have

- $\mathsf{Ind}\,\mathbb{Q}$ is Weihrauch incomparable with $\mathsf{C}_\mathbb{N}$ (Theorem 4.1), which can be viewed as the problem which finds a monochromatic rational interval given a coloring for which there is one.
- $\mathsf{Ind}\,\mathbb{Q}_k$ cannot be solved by any problem having a c.e. approximation (Theorem 4.5), in a precise sense given by Definition 4.4 which encompasses $\mathsf{RT}^1_\mathbb{N}$ and any problem with finite computable codomain, among other things.

The hierarchy of Weihrauch reductions (and nonreductions) relating versions of $\mathsf{TC}^k_\mathbb{N}$, $\mathsf{Ind}\,\mathbb{Q}_k$, and RT^1_k is shown in Figure 1 below. Next, $\mathsf{Ind}\,\mathscr{E}$ turns out to be closely related to but distinct from Ramsey's theorem for pairs:

- The Weihrauch degree of $\mathsf{Ind}\,\mathscr{E}_k$ is strictly between those of SRT^2_k and RT^2_k for all $k \geq 2$;
- In particular, $\mathsf{Ind}\,\mathscr{E}_2$ is not Weihrauch reducible to $\mathsf{SRT}^2_\mathbb{N}$ and RT^2_2 is not computably reducible to $\mathsf{Ind}\,\mathscr{E}_\mathbb{N}$ (Theorem 5.1).

The Weihrauch reductions between these principles are displayed in Figure 4.

Before getting to our main results in Sections 4 and 5, we define all the reducibilities under consideration in Section 2 as well as the main problems used as benchmarks. In Section 3 we comment on the role of uniform computable categoricity in the present investigation and collect several properties of indivisibility problems in general. Finally, in Section 6 we briefly mention some directions for future research on $\mathsf{Ind}\,\mathbb{Q}$ and $\mathsf{Ind}\,\mathscr{E}$.

2. Preliminaries

The standard reference on reverse mathematics is Simpson [Sim09]. The books by Hirschfeldt [Hir15] and by Dzhafarov and Mummert [DM22] are more computability-oriented treatments of the subject, the latter discussing in detail the reducibilities we employ. Our notation and conventions with respect to computability theory are standard and can be found in any modern textbook, e.g., Soare [Soa16]. For any set X and $n \in \mathbb{N}$, $X \upharpoonright n$ is the truncation of X up through its first n bits. If $\sigma \in \mathbb{N}^{<\mathbb{N}}$, then $[\sigma]$ denotes the set of finite or infinite strings extending σ. We will always identify $k \in \mathbb{N}$ with the set $\{0, \ldots, k-1\}$. Write W_e for the eth c.e. set with respect to some computable enumeration of Turing machines, and W_e^X for the eth X-c.e. set.

Definition 2.1. Let P and Q be problems. Then

- P is *computably reducible* to Q, written $P \leq_{\mathrm{c}} Q$, if each $p \in \operatorname{dom} P$ computes some $x \in \operatorname{dom} Q$ such that for every $q \in Q(x)$, $p \oplus q$ computes an element of $P(p)$. If we require q by itself to compute an element of $P(p)$, then P is *strongly computably reducible* to Q, written $P \leq_{\mathrm{sc}} Q$.
- P is *Weihrauch reducible* to Q, written $P \leq_{\mathrm{W}} Q$, if there are Turing functionals Δ and Ψ such that if $p \in \operatorname{dom} P$, then $\Delta^p \in \operatorname{dom} Q$; and for any $q \in Q(\Delta^p)$, we have $\Psi^{p \oplus q} \in P(p)$. If we require Ψ to only have oracle access to q, then P is *strongly Weihrauch reducible* to Q, written $P \leq_{\mathrm{sW}} Q$.

Δ and Ψ are sometimes called the forward and return functionals, respectively. We have $P \leq_{\mathrm{sW}} Q \implies P \leq_{\mathrm{W}} Q \implies P \leq_{\mathrm{c}} Q$ and $P \leq_{\mathrm{sW}} Q \implies P \leq_{\mathrm{sc}} Q \implies P \leq_{\mathrm{c}} Q$, but none of these implications reverse and there is no logical relationship between $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sc}}$. All of these reducibilities also imply that Q implies P in ω-models of the weak logical system RCA_0, i.e., models with the standard natural numbers. Write $P|_{\Box}Q$ if P and Q are incomparable under $\leq_{\Box}$, and $P \equiv_{\Box} Q$ if $P \leq_{\Box} Q$ and $Q \leq_{\Box} P$, for $\Box \in \{\mathrm{c}, \mathrm{sc}, \mathrm{W}, \mathrm{sW}\}$. In the latter case P and Q are said to be equivalent or to have the same degree with respect to $\leq_{\Box}$. See [HJ16, DM22] for details on these and other reducibilities, and [BGP21] for a good recent survey of Weihrauch reducibility with a large bibliography and historical remarks. The reducibilities above can be defined between problems on any represented spaces, but we do not need this level of generality. More information can again be found in [BGP21], and we largely follow the latter treatment here.

There are a few algebraic operations on problems which we will have occasion to use. Let P and Q be problems. Then

- $P \times Q$ is the *parallel product* of P and Q, defined by $\operatorname{dom}(P \times Q) = \operatorname{dom} P \times \operatorname{dom} Q$, and $(P \times Q)(p, q) = P(p) \times Q(q)$.
- P^n is the n-fold parallel product of P with itself.
- P^* is the *finite parallelization* of P, which solves P^n for any n. Instances include the data n.
- $\hat{P}$ is the *parallelization* of P. Instances are sequences $\langle p_0, p_1, p_2, \ldots \rangle \in \operatorname{dom}(P)^{\mathbb{N}}$, and the set of solutions to this instance is the Cartesian product $\prod_{i \in \mathbb{N}} P(p_i)$.
- $P \sqcup Q$ is the *coproduct* of P and Q, with $\operatorname{dom}(P \sqcup Q) = \operatorname{dom} P \sqcup \operatorname{dom} Q$, and $(P \sqcup Q)(0, p) = \{0\} \times P(p)$ and $(P \sqcup Q)(1, q) = \{1\} \times Q(q)$. So $P \sqcup Q$ is the problem which is capable of solving any instance of P or of Q, but only one at a time.
- P' is the *jump* of P. An instance of P' is a sequence $\langle p_0, p_1, p_2, \ldots \rangle$ of reals converging to some $p \in \operatorname{dom} P$ in the Baire space topology (equivalently, converging entrywise). The solutions to this instance are just the elements of $P(p)$. So P' answers the same question about $p \in \operatorname{dom} P$ that P does, but only has access to a limit representation of p. We also set $P^{(n+1)} = (P^{(n)})'$ with $P^{(0)} = P$.
- $P * Q$ is the *compositional product* of P and Q. This can be characterized intuitively as the strongest problem under $\leq_{\mathrm{W}}$ obtainable as the composition of f and g, ranging over all $f \leq_{\mathrm{W}} P$ and $g \leq_{\mathrm{W}} Q$. (A more precise definition of a representative of the degree of $P * Q$ is given in Section 3 below.)

The following "benchmark" problems will be used as a basis for comparison with the problems we study:

- LPO, the *limited principle of omniscience*, has $\operatorname{dom} \mathsf{LPO} = \mathbb{N}^{\mathbb{N}}$, with $\mathsf{LPO}(0^{\mathbb{N}}) = 0$ and $\mathsf{LPO}(p) = 1$ otherwise. $\mathsf{LPO}^{(n)}$ can be thought of as answering a single Σ^0_{n+1} question.

- lim maps a convergent sequence of reals $\langle p_0, p_1, p_2, \ldots\rangle$ to its limit.
- $\mathsf{C}_\mathbb{N}$, closed choice on $\mathbb{N}$, outputs an element of a nonempty set $A \subseteq \mathbb{N}$ given an enumeration of its complement.
- $\mathsf{TC}_\mathbb{N}$, the totalization of $\mathsf{C}_\mathbb{N}$, extends $\mathsf{C}_\mathbb{N}$ by allowing $A = \emptyset$ and outputting any number in this case.
- RT^n_k, Ramsey's theorem for n-tuples and k colors, has instances $c\colon [\mathbb{N}]^n \to k$, and solutions to c are (characteristic functions of) infinite c-homogeneous sets. Here $[X]^n$ is the set of n-element subsets of X, and a set $X \subseteq \mathbb{N}$ is *homogeneous* for c if X is infinite and c is monochromatic on $[X]^n$.
- SRT^2_k, stable Ramsey's theorem for pairs, is the restriction of RT^2_k to *stable colorings*, i.e., colorings c such that $\lim_m c\{n, m\}$ exists for all n.
- $\mathsf{RT}^n_\mathbb{N}$ has instances $c \in \bigcup_{k\in\mathbb{N}} \operatorname{dom} \mathsf{RT}^n_k$. Solutions to c are again infinite c-homogeneous sets. Notice that in this formulation, the number of colors used is not included as part of an instance. $\mathsf{SRT}^2_\mathbb{N}$ is defined similarly.
- For $k \in \mathbb{N} \cup \{\mathbb{N}\}$, cRT^n_k is the "color version" of RT^n_k, which only outputs the colors of RT^n_k-solutions. We have $\mathsf{RT}^1_k \equiv_\mathrm{W} \mathsf{cRT}^1_k$ for all k, since the color can be used to compute the set of points of that color and vice versa.
- RT^n_+ is $\bigsqcup_{k\in\mathbb{N}} \mathsf{RT}^n_k$, and cRT^n_+ and SRT^2_+ are defined similarly.

Clearly $\mathsf{RT}^n_2 \leq_\mathrm{sW} \mathsf{RT}^n_3 \leq_\mathrm{sW} \ldots \leq_\mathrm{sW} \mathsf{RT}^n_+ \leq_\mathrm{sW} \mathsf{RT}^n_\mathbb{N}$ for all n, and similarly for SRT^2_k. Also $\mathsf{Ind}\,\mathcal{S}_2 \leq_\mathrm{sW} \mathsf{Ind}\,\mathcal{S}_3 \leq_\mathrm{sW} \ldots \leq_\mathrm{sW} \mathsf{Ind}\,\mathcal{S} \leq_\mathrm{sW} \mathsf{Ind}\,\mathcal{S}_\mathbb{N}$ for any indivisible $\mathcal{S}$.

3. Uniform computable categoricity

Recall that a computable structure $\mathcal{A}$ is uniformly computably categorical (u.c.c.) if there is a Turing functional which computes an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ given the atomic diagram of any presentation $\mathcal{B}$ of $\mathcal{A}$. If $\mathcal{S}$ is indivisible and u.c.c., then from the point of view of $\leq_\mathrm{W}$ we can as a convention regard the instances of $\mathsf{Ind}\,\mathcal{S}$ as only including the data k and $c\colon \mathbb{N} \to k$, since in a reduction, the functionals Δ and Ψ can just build in the translations between any given presentation $\mathcal{A}$ and some fixed computable presentation of $\mathcal{S}$. Instances of $\mathsf{Ind}\,\mathcal{S}_k$ and $\mathsf{Ind}\,\mathcal{S}_\mathbb{N}$ are viewed simply as colorings c of $\mathbb{N}$, which is justified as long as we choose a computable reference presentation $\mathcal{S}$. One can state this more formally as

Proposition 3.1. *If $\mathcal{S}$ is indivisible and u.c.c., then $\mathsf{Ind}\,\mathcal{S}_k \equiv_\mathrm{W} P_k$ where P_k is the restriction of $\mathsf{Ind}\,\mathcal{S}_k$ to instances of the form $\langle \mathcal{S}, c\rangle$. Analogous statements hold for $\mathsf{Ind}\,\mathcal{S}$ and $\mathsf{Ind}\,\mathcal{S}_\mathbb{N}$.* □

These conventions are not a priori applicable when considering $\leq_\mathrm{sW}$ (or $\leq_\mathrm{sc}$), because the return functional Ψ could need oracle access to the presentation $\mathcal{A}$ in order to translate back to a solution of $\langle \mathcal{A}, c, k\rangle$ if Δ modified $\mathcal{A}$. A version of Proposition 3.1 with $\equiv_\mathrm{W}$ replaced by $\equiv_\mathrm{c}$ still holds if $\mathcal{S}$ is merely relatively computably categorical, i.e., if the functional computing the isomorphism from $\mathcal{B}$ to $\mathcal{A}$ (as above) is allowed to depend on $\mathcal{B}$.[1] Proposition 3.2 below also continues to hold for relatively computably categorical $\mathcal{S}$, so long as $\equiv_\mathrm{W}$ is changed to $\equiv_\mathrm{c}$ everywhere.

[1]Technically, what we call "uniform computable categoricity" should really be called uniform relative computable categoricity. The standard definition of uniform computable categoricity of a computable structure $\mathcal{A}$ is that there is a Turing functional which, given any *computable* copy $\mathcal{B}$ of $\mathcal{A}$ as oracle, computes an isomorphism from $\mathcal{B}$ to $\mathcal{A}$. However, this is equivalent to our definition by a theorem of Ventsov [Mon21, Theorem III.18]. Relative and plain computable categoricity differ; see Chapter VIII of [Mon21] for details.

Both of the structures we focus on in this paper are uniformly computably categorical and so we will follow the above convention without further comment. That $\mathbb{Q}$ is u.c.c. follows from the classical back-and-forth argument, which is effective. To see that $\mathscr{E}$ is u.c.c., for any presentation $\mathcal{A}$ of $\mathscr{E}$, decompose any $n \in A$ as a pair $\langle x, y\rangle_{\mathcal{A}}$ so that n is the yth element of the xth distinct equivalence class, in order of discovery within the atomic diagram of $\mathcal{A}$. Then if also $m = \langle z, w\rangle_{\mathcal{A}}$, we have that n and m are equivalent iff $x = z$, and $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ is uniformly $\mathcal{A}$-computable. (This definition does not uniquely specify $\langle\cdot,\cdot\rangle_{\mathcal{A}}$, but one can make some canonical choice.) If we take $\mathscr{E}$ to be computable and $\mathcal{A}$ is a given copy of $\mathscr{E}$, without loss of generality with $E = A = \mathbb{N}$, then the map $\langle x, y\rangle_{\mathcal{A}} \mapsto \langle x, y\rangle_{\mathscr{E}}$ is a uniformly $\mathcal{A}$-computable isomorphism between $\mathcal{A}$ and $\mathscr{E}$.

The problems corresponding to u.c.c. indivisible structures enjoy several nice properties. For example, it is not hard to see that the indivisibility and strong indivisibility problems of such a structure must be Weihrauch equivalent. There is also the following easy observation:

Proposition 3.2. *If* $\mathcal{S}$ *is indivisible and u.c.c., then* $\mathsf{Ind}\,\mathcal{S} \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}^*$*. Similarly for* $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$*.*

Proof. It suffices to show $\mathsf{Ind}\,\mathcal{S}^* \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}$. Since $\mathcal{S}$ is u.c.c., as noted above, we can assume each instance of $\mathsf{Ind}\,\mathcal{S}$ consists only of a coloring together with a number of colors. Then suppose we are given instances $\langle c_0, k_0\rangle, \ldots, \langle c_n, k_n\rangle$ of $\mathsf{Ind}\,\mathcal{S}$ in parallel, with $c_i \colon \mathbb{N} \to k_i$ for each i. Let p_i be the ith prime number and define $d \colon \mathbb{N} \to \prod_i p_i^{k_i+1}$ by $d(x) = \prod_i p_i^{c_i(x)+1}$. If H is a solution to d with color j, then $j = \prod_i p_i^{j_i+1}$ for some numbers $j_0 < k_0, \ldots, j_n < k_n$. Hence $c_i(H) = j_i$ for each $i = 0, \ldots, n$, and H is simultaneously a solution to $\langle c_i, k_i\rangle$ for all i. It is clear that the argument works unchanged for $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$. □

Next we have the following result which extends Proposition 62 of [PPS24] from $\mathbb{N}$ to any indivisible, u.c.c. $\mathcal{S}$. Our argument is really only a slight adaptation of the original, but we give it in detail for completeness. It will be useful for the proof to use the representative $f \star g$ of the Weihrauch degree of $f * g$, defined by

$$(f \star g)(x, y) = \langle \mathrm{id} \times f\rangle \circ \Phi_x \circ g(y),$$

where id is the identity map on $\mathbb{N}^{\mathbb{N}}$ and Φ is a universal functional. (See [BGP21, Definition 11.5.3].)

Proposition 3.3. *For any indivisible* $\mathcal{S}$*, we have* $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{Ind}\,\mathcal{S} * \mathsf{C}_{\mathbb{N}}$*. If* $\mathcal{S}$ *is u.c.c., then also* $\mathsf{Ind}\,\mathcal{S} * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$*.*

Proof, after [PPS24]. For the first statement, let $\mathcal{S}$ be an arbitrary indivisible structure and let $\langle \mathcal{A}, c\rangle$ be a given instance of $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$. Build a $\mathsf{C}_{\mathbb{N}}$-instance by, whenever a new color is seen in c, enumerating all numbers less than that color. One can use a $\mathsf{C}_{\mathbb{N}}$-solution k together with $\langle \mathcal{A}, c\rangle$ (which can be encoded as part of the program to be run by Φ) to compute the instance $\langle \mathcal{A}, c, k\rangle$ of $\mathsf{Ind}\,\mathcal{S}$, and a solution to this instance is also a solution to $\langle \mathcal{A}, c\rangle$.

To prove the second statement, let Bound be the problem that outputs an upper bound on an enumeration of a finite set. This is Weihrauch equivalent to $\mathsf{C}_{\mathbb{N}}$ and it will be convenient to show that $\mathsf{Ind}\,\mathcal{S} \star \mathsf{Bound} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$. Let (x, Y) be an instance of $\mathsf{Ind}\,\mathcal{S} \star \mathsf{Bound}$, so Y is a finite set represented by an (infinite) enumeration. Since $\mathcal{S}$ is u.c.c., we can treat Φ_x in its second component as computing only a coloring with an upper bound on its range. For each $i \in \mathbb{N}$, let p_i be the ith prime number as before. Build $d \in \mathrm{dom}(\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}})$ as follows: find a number i_0 such that $\Phi_x(i_0)$ outputs, in addition to a partial coloring c_0 of $\mathbb{N} = \mathrm{dom}\,\mathcal{S}$, an upper bound k_0 on the range of c_0. This i_0 is an initial guess for an element of $\mathsf{Bound}(Y)$.

Set $d(n) = p_{i_0}^{c_0(n)}$ for all n for which we see $c_0(n)$ defined by $\Phi_x(i_0)$. In general, if i_s has been found, let $d(n) = p_{i_s}^{c_s(n)}$ whenever $c_s(n)$ is defined and n had not been colored at a previous stage. If numbers are enumerated into Y so that $\max Y \geq i_s$, find an $i_{s+1} > \max Y$ such that $\Phi_x(i_{s+1})$ outputs a number k_{s+1}; such an i_{s+1} must exist, so we can hold off on extending d until k_{s+1} appears. When it does, continue with $d(n) = p_{i_{s+1}}^{c_{s+1}(n)}$ for all n which had not been previously colored and for which $c_{s+1}(n)$ is defined.

Eventually this process stabilizes at some i_ℓ, since Y is finite, and once it does $\Phi_x(i_\ell)$ must produce a total coloring. Then any solution H of d has color $p_{i_\ell}^a$ for some $a < k_\ell$. By rerunning the procedure in the last paragraph, the return functional can recover d, hence find the color of H, and from that learn i_ℓ and output the first component of $\Phi_x(i_\ell)$ to satisfy id. Finally, H is in fact a solution to the second component of $\Phi_x(i_\ell)$, because it can only include points colored after i_ℓ stabilizes and so the fact that d and the coloring computed by $\Phi_x(i_\ell)$ differ on finitely many other points is of no consequence. □

If $\mathcal{S}$ is not u.c.c., then d could still be built to eventually agree (up to taking prime powers) with the correct coloring c_ℓ, so that it has the same solutions. But one would also need to somehow copy the atomic diagram computed by $\Phi_x(i_s)$ to produce an instance of $\mathsf{Ind}\,\mathcal{S}_\mathbb{N}$, and there is no reason something eventually agreeing with a correct presentation should be a presentation itself at all, never mind one for which d has the same solutions. On the other hand, exactly the same proof serves to show that $\mathsf{RT}^n_\mathbb{N} \equiv_\mathrm{W} \mathsf{RT}^n_+ * \mathsf{C}_\mathbb{N}$ for all n and that $\mathsf{SRT}^2_\mathbb{N} \equiv_\mathrm{W} \mathsf{SRT}^2_+ * \mathsf{C}_\mathbb{N}$.

The nonuniform situation is considerably simpler. $P * \mathsf{C}_\mathbb{N} \equiv_\mathrm{c} P$ for any problem P since the solutions to $\mathsf{C}_\mathbb{N}$ are just single numbers, so the problems $\mathsf{Ind}\,\mathcal{S}$, $\mathsf{Ind}\,\mathcal{S}_\mathbb{N}$, and $\mathsf{Ind}\,\mathcal{S} * \mathsf{C}_\mathbb{N}$ collapse under $\leq_\mathrm{c}$, and indeed

Proposition 3.4. *For any indivisible $\mathcal{S}$, we have $\mathsf{Ind}\,\mathcal{S} \equiv_\mathrm{sc} \mathsf{Ind}\,\mathcal{S}_\mathbb{N}$.*

Proof. $\mathsf{Ind}\,\mathcal{S} \leq_\mathrm{sc} \mathsf{Ind}\,\mathcal{S}_\mathbb{N}$ is always true, and $\mathsf{Ind}\,\mathcal{S}_\mathbb{N} \leq_\mathrm{sc} \mathsf{Ind}\,\mathcal{S}$ follows because one can nonuniformly append the numbers of colors used by an $\mathsf{Ind}\,\mathcal{S}_\mathbb{N}$-instance in order to get an $\mathsf{Ind}\,\mathcal{S}$-instance with the same solutions. □

We now discuss one more general property which will be useful later. A problem f is called a *fractal* if there is an $F\colon \subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ with $f \equiv_\mathrm{W} F$ such that $F\restriction_A \equiv_\mathrm{W} F$ for every clopen A with $A \cap \operatorname{dom} F \neq \emptyset$, and f is a *total* or *closed fractal* if F can additionally be taken with domain $\mathbb{N}^\mathbb{N}$ [BGP21, Definition 11.4.10]. Intuitively, being a fractal means that no matter how far one zooms into the domain of f, its full power is always present.

We do not expect $\mathsf{Ind}\,\mathcal{S}_k$ to be a fractal even for an arbitrary u.c.c. structure, but it turns out to be one for both $\mathcal{S} = \mathbb{Q}$ and $\mathscr{E}$ because both structures have the property that deleting any finite set produces an isomorphic substructure. This observation can be made more general by invoking the notion of finite tolerance, originally introduced in [DDH$^+$16] (see [BGP21, Definition 11.4.5]). A problem f is *finitely tolerant* if there is a computable functional Φ such that for any $x \in \operatorname{dom} f$, for any $y \in f(x)$, we have that $\Phi(y, n)$ computes a solution to $f(z)$ for any instance z such that $z(i) = x(i)$ for all $i \geq n$. Thus $\mathsf{Ind}\,\mathbb{Q}_k$ and $\mathsf{Ind}\,\mathscr{E}_k$ are finitely tolerant: in each case, given a solution H to a coloring c along with n, Φ can just delete the first n elements from H to produce a solution to any coloring d agreeing with c after the first n bits. The same trick applies to any structure which is isomorphic to every cofinite substructure. (Not every indivisible structure has the latter property; consider $(\mathbb{Q} + 1, <)$.)

Proposition 3.5.

(i) *If* $\mathcal{S}$ *is u.c.c. and* $\mathsf{Ind}\,\mathcal{S}_k$ *is finitely tolerant, then* $\mathsf{Ind}\,\mathcal{S}_k$ *is a total fractal.*
(ii) $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ *is a fractal for any u.c.c.* $\mathcal{S}$.

Proof.

(i) If x is any real, define $\lfloor x \rfloor_k$ by setting $\lfloor x \rfloor_k(i) = \min\{k-1, x(i)\}$ for each i. Then let $f\colon \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be given by $f(x) = \mathsf{Ind}\,\mathcal{S}_k(\lfloor x \rfloor_k)$. Viewing the instances of $\mathsf{Ind}\,\mathcal{S}_k$ as colorings of a fixed computable presentation, it is clear that $f \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_k$: on the one hand, $\mathsf{Ind}\,\mathcal{S}_k$ is a subproblem of f, and on the other hand, we can reduce f to $\mathsf{Ind}\,\mathcal{S}_k$ by computing $\lfloor x \rfloor_k$ from a given x and feeding it into $\mathsf{Ind}\,\mathcal{S}_k$. Therefore the latter is Weihrauch equivalent to a total problem. Notice $f(x) = f(\lfloor x \rfloor_k)$ for all x. (So far we have not used the assumption of finite tolerance.)

We now show that this f is a fractal by reducing it to its restriction to a given cylinder. Let Φ witness finite tolerance of $\mathsf{Ind}\,\mathcal{S}_k$. Given a finite string σ and any $c \in \mathbb{N}^{\mathbb{N}}$, pass to $\lfloor c \rfloor_k$ and overwrite the first $|\sigma|$ bits of the latter string with those of σ to produce an instance d of $f \restriction_{[\sigma]}$. If H is an f-solution to d, then it is also an $\mathsf{Ind}\,\mathcal{S}_k$-solution to d, of course. Since d agrees with $\lfloor c \rfloor_k$ past the first $|\sigma|$ bits, $\Phi(H, |\sigma|)$ is an $\mathsf{Ind}\,\mathcal{S}_k$-solution to $\lfloor c \rfloor_k$, hence an f-solution to the same, and hence also an f-solution to the original c.

(ii) It suffices to show $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \restriction_{[\sigma]}$ for every $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Let n be the largest entry of σ. Given any bounded coloring c of S, produce a new coloring $d \in [\sigma]$ by replacing the first $|\sigma|$ bits of c with σ and adding $n+1$ to the remaining bits. Then every d-solution is also a c-solution. □

It is known that if f is a fractal and $f \leq_{\mathrm{W}} \bigsqcup_{i \in \mathbb{N}} g_i$ for any problems g_i, then $f \leq_{\mathrm{W}} g_i$ for some i [BdBP12, Lemma 5.5]. Additionally, if f is a total fractal and $f \leq_{\mathrm{W}} g * \mathsf{C}_{\mathbb{N}}$ for any g, then $f \leq_{\mathrm{W}} g$ [LRP13, Theorem 2.4]. Thus $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ is not in general a total fractal: if it were, Proposition 3.3 would imply $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}$, and hence $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_k$ for some k. This fails for $\mathscr{E}$ by Corollary 5.4 and for $\mathbb{Q}$ by the discussion in Section 4.1, which shows that $\mathsf{Ind}\,\mathbb{Q}_{k+1} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$. (It seems unlikely that it could happen for any $\mathcal{S}$, though we do not have a proof it does not.) For the same reason, neither $\mathsf{Ind}\,\mathbb{Q}$ nor $\mathsf{Ind}\,\mathscr{E}$ is a fractal. One may compare this situation to that of the $\mathsf{Shuffle}$ family of problems, which were studied in [PPS24] and whose definition is given in the next section. As per the discussion in Section 6 of that paper, the versions of those problems with a fixed number of colors are all total fractals; the versions where instances have no information on the number of colors are fractals, but not total fractals; and the versions where instances have a given but not fixed number of colors are not fractals at all.

The condition of uniform computable categoricity in Propositions 3.2, 3.3, and 3.5 is sufficient but not necessary, since they hold for $\mathsf{Ind}\,\mathbb{N} \equiv_{\mathrm{W}} \mathsf{RT}^1_+$ while $(\mathbb{N}, <)$ is not computably categorical. (It is perhaps worth noting that $\mathsf{Ind}\,(\mathbb{N}, <)$ is Weihrauch equivalent to $\mathsf{Ind}\,(\mathbb{N}, \emptyset)$, the structure consisting of a countably infinite set in the empty language, and the latter structure *is* u.c.c.) We do not know a characterization of the structures $\mathcal{S}$ for which the above results hold. A good source of (counter)examples could be the class of nonscattered (countable) linear orders, all of which are indivisible as a consequence of the fact that every countable linear order embeds into $\mathbb{Q}$. The only infinite u.c.c. linear order is $\mathbb{Q}$ itself (see e.g. [Mon21, Example III.4]), but the indivisibility problems of many other linear orders turn out to have exactly the same computational strength as those of $\mathbb{Q}$. A linear order $\mathcal{L}$ is called *strongly η-like* if there is a number n such that $\mathcal{L}$ can be built from $\mathbb{Q}$ by replacing every element by a chain of between one and n elements. (Here η is the order type of $\mathbb{Q}$. See for instance [Dow98, §4] or [Hir15, §10.1] for a discussion of this notion.) Then we have

Proposition 3.6. *If $\mathcal{M}$ is a strongly η-like computable linear order, and $\mathcal{L} = (m_1+\mathcal{M}+m_2, <)$ for some $m_1, m_2 \in \mathbb{N}$, then $\mathsf{Ind}\,\mathcal{L}_k \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$. Similarly for $\mathsf{Ind}\,\mathcal{L}$ and $\mathsf{Ind}\,\mathcal{L}_{\mathbb{N}}$.*

Proof. [$\leq_{\mathrm{W}}$] Given an instance $\langle \mathcal{A}, c\rangle$ of $\mathsf{Ind}\,\mathcal{L}_k$, we will produce a subcopy of $\mathbb{Q}$ as follows. Let n witness that $\mathcal{M}$ is strongly η-like. Read far enough in the atomic diagram of $\mathcal{A}$ to find two points $x_0 < x_1$ such that there are more than m_1 points below x_0, more than m_2 above x_1, and more than n between x_0 and x_1. At each subsequent stage, for every $x_i < x_j$ found so far, search for an $x_h \in (x_i, x_j)$ such that (x_i, x_h) and (x_h, x_j) each contain more than n points. This criterion ensures that x_h cannot be finitely apart from either x_i or x_j, so that both (x_i, x_h) and (x_h, x_j) contain densely ordered sets. Hence it is always possible to find a suitable x_h at every step, and $X = \{x_2, x_3, \dots\}$ is a uniformly $\mathcal{A}$-computable set which induces a substructure $\mathcal{X} \simeq \mathbb{Q}$. Then $\langle \mathcal{X}, c\rangle$ is an instance of $\mathsf{Ind}\,\mathbb{Q}_k$; if H is a solution to this instance, then by density of H, we can use a greedy algorithm to $\mathcal{A}$-compute a subcopy of $\mathcal{A}$ within H. This subcopy is a solution to the original instance $\langle \mathcal{A}, c\rangle$ of $\mathsf{Ind}\,\mathcal{L}_k$. (This direction of the reduction does not require $\mathcal{M}$ to have a computable copy.)

[$\geq_{\mathrm{W}}$] The proof is essentially the same as the above, but in reverse. Let $\langle \mathcal{A}, c\rangle \in \mathrm{dom}(\mathsf{Ind}\,\mathbb{Q}_k)$. Build an instance of $\mathsf{Ind}\,\mathcal{L}_k$ by producing a subcopy $\mathcal{B}$ of $\mathcal{L}$ inside $\mathcal{A}$ and using c as-is. (This step is the only place where we use that $\mathcal{L}$ is computable, since the forward functional in this reduction does not have oracle access to a presentation of $\mathcal{L}$.) If H is an $\mathsf{Ind}\,\mathcal{L}_k$-solution to $\langle \mathcal{B}, c\rangle$, then one can carry out the procedure described above to find X inside H, and X will be a solution to $\langle \mathcal{A}, c\rangle$. □

Consequently, the conclusions of Propositions 3.2, 3.3, and 3.5 hold for any such $\mathcal{L}$.

4. The rational numbers

4.1. **Prior related work.** $\mathsf{Ind}\,\mathbb{Q}$ is exactly the problem form of the reverse-mathematical principle ER^1 studied by Frittaion and Patey in [FP17]. There the authors show, among other things, that the implication $\mathsf{ER}^1 \to \mathsf{RT}^1_+$ over RCA_0 is strict. This has a kind of uniform counterpart in our Corollary 4.7(i), since trivially $\mathsf{RT}^1_+ \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}$ (and $\mathsf{RT}^1_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$) for any indivisible $\mathcal{S}$ with a computable presentation, and this is strict if $\mathcal{S} = \mathbb{Q}$. But separations over RCA_0 have no direct bearing in the present setting: both $\mathbb{Q}$ and $\mathbb{N}$ are computably indivisible and hence both $\mathsf{Ind}\,\mathbb{Q}$ and RT^1_+ hold in ω-models of RCA_0, indeed are computably equivalent to the identity map on Baire space. To obtain meaningful distinctions between them one must pass to a stronger reducibility, and so we focus on the uniform content of $\mathsf{Ind}\,\mathbb{Q}$ via its Weihrauch degree.

The Weihrauch degrees of a family of problems related to $\mathsf{Ind}\,\mathbb{Q}$ have recently been studied by Pauly, Pradic, and Soldà [PPS24]. In their terminology, if c is any coloring of $\mathbb{Q}$, then an open interval $I \subseteq \mathbb{Q}$ is a *c-shuffle* if for every color occurring in I, the set of points of that color is dense in I. They then investigate several corresponding families of problems:

- $\mathsf{Shuffle}$ has instances (k, c), where $k \in \mathbb{N}$ and $c\colon \mathbb{Q} \to k$. If I is a (code for a) rational interval and $C \subseteq k$, then $(I, C) \in \mathsf{Shuffle}(k, c)$ iff I is a c-shuffle with exactly the colors of C.
- $\mathsf{iShuffle}$ is the weakening of $\mathsf{Shuffle}$ which, for an instance (k, c), returns an interval I such that there exists $C \subseteq k$ with $(I, C) \in \mathsf{Shuffle}(k, c)$.

- $\mathsf{cShuffle}$ is the weakening of $\mathsf{Shuffle}$ which, for an instance (k, c), returns a $C \subseteq k$ such that there is an I with $(I, C) \in \mathsf{Shuffle}(k, c)$.
- $(\eta)^1_{<\infty}$, which had previously been studied in [FP17] as a reverse-mathematical principle, is a different weakening of $\mathsf{Shuffle}$ which, for an instance (k, c), returns an interval I and a single color $n < k$ such that the points of c-color n are dense in I.

$\mathsf{Shuffle}_k$, $\mathsf{iShuffle}_k$, and $\mathsf{cShuffle}_k$ are defined to be the restrictions of the first three problems to instances of the form (k, c). One can also define $\mathsf{Shuffle}_{\mathbb{N}}$, $\mathsf{iShuffle}_{\mathbb{N}}$, and $\mathsf{cShuffle}_{\mathbb{N}}$ in a natural way. The authors show that $\mathsf{cShuffle} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$, $\mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \equiv_{\mathrm{W}} (\eta)^1_{<\infty} \equiv_{\mathrm{W}} \mathsf{i}(\eta)^1_{<\infty}$ where the latter problem is the version of $(\eta)^1_{<\infty}$ which only returns an interval I, and $\mathsf{Shuffle} \equiv_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}^*_{\mathbb{N}}$. More specifically, they show $\mathsf{iShuffle}_k \equiv_{\mathrm{W}} \mathsf{TC}^{k-1}_{\mathbb{N}}$, the $(k-1)$-fold parallel product of $\mathsf{TC}_{\mathbb{N}}$. (They also show that $\mathsf{cShuffle}_k \leq_{\mathrm{W}} (\mathsf{LPO}')^{2^k-2}$. The reverse direction of the equivalence between $\mathsf{cShuffle}$ and $(\mathsf{LPO}')^*$ is established by showing that $\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{cShuffle}$ and $\mathsf{cShuffle} \equiv_{\mathrm{W}} \mathsf{cShuffle}^*$. The precise relationship between number of colors and number of parallel instances of LPO' is left open.) As part of their investigation, they establish further that $\mathsf{iShuffle}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{iShuffle} * \mathsf{C}_{\mathbb{N}}$ (Proposition 63), and also that $\mathsf{cRT}^1_{k+1} \leq_{\mathrm{W}} \mathsf{TC}^m_{\mathbb{N}}$ if and only if $k \leq m$ (Theorem 10). Since $\mathsf{RT}^1_{\mathbb{N}}$ is a fractal, it follows that $\mathsf{iShuffle}$ cannot solve $\mathsf{RT}^1_{\mathbb{N}}$: if it could, then we would have $\mathsf{RT}^1_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{cRT}^1_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{TC}^k_{\mathbb{N}}$ for some k, a contradiction.

It is clear that $\mathsf{Ind}\,\mathbb{Q}_k \leq_{\mathrm{W}} \mathsf{iShuffle}_k$, $\mathsf{Ind}\,\mathbb{Q} \leq_{\mathrm{W}} \mathsf{iShuffle}$, and $\mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{iShuffle}_{\mathbb{N}}$, since for any color i found in a shuffle, the set of points of that color in the shuffle is isomorphic to $\mathbb{Q}$. As $\mathsf{iShuffle}_k$ cannot solve cRT^1_{k+1}, it follows that $\mathsf{Ind}\,\mathbb{Q}_{k+1} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$. Similarly, the fact that $\mathsf{RT}^1_{\mathbb{N}}$ is below $\mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}}$ but not $\mathsf{iShuffle}$ implies that $\mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{iShuffle}$ and in particular $\mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$.

After a draft of this article appeared, the author was made aware of contemporaneous work by Dzhafarov, Solomon, and Valenti [DSV23] concerning the *tree pigeonhole principle* TT^1_+. This states that given any coloring of $2^{<\mathbb{N}}$ with bounded range, there is an infinite monochromatic subset H isomorphic to $2^{<\mathbb{N}}$ as a partial order. Problems TT^1_k and $\mathsf{TT}^1_{\mathbb{N}}$ are defined analogously. For any such H, we have $(H, <_{KB}) \simeq (\mathbb{Q}+1, <)$ where $<_{KB}$ is the Kleene-Brouwer linear order. Then $(2^{<\mathbb{N}}, <_{KB})$ can be viewed as a certain computable presentation of $\mathbb{Q}+1$, which becomes a presentation of $\mathbb{Q}$ once the empty string ε (i.e., the root node) is deleted. It follows that $\mathsf{TT}^1_+ \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$, and similarly for TT^1_k and $\mathsf{TT}^1_{\mathbb{N}}$. To prove this, note that by Proposition 3.6, it suffices to show that $\mathsf{Ind}\,\mathbb{Q} \leq_{\mathrm{W}} \mathsf{TT}^1_+ \leq_{\mathrm{W}} \mathsf{Ind}\,(\mathbb{Q}+1)$. For the first reduction, given an instance $\langle \mathcal{A}, c\rangle$ of $\mathsf{Ind}\,\mathbb{Q}$, by uniform computable categoricity we may as well assume $(\mathcal{A}, <_{\mathcal{A}}) = (2^{<\mathbb{N}} \setminus \{\varepsilon\}, <_{KB})$. Then build an instance of TT^1_+ by coloring ε arbitrarily and otherwise copying c as-is. If H is a solution to this instance, then $(H, <_{KB}) \simeq (2^{<\mathbb{N}}, <_{KB}) \simeq (\mathbb{Q}+1, <)$, so in order to find a subcopy of $\mathbb{Q}$, one can pick any $\sigma \in H$ and restrict H to the set of elements strictly extending σ. As for the reduction of TT^1_+ to $\mathsf{Ind}\,(\mathbb{Q}+1)$, given a coloring c of $2^{<\mathbb{N}}$, create an instance of $\mathsf{Ind}\,(\mathbb{Q}+1)$ by using the presentation $(2^{<\mathbb{N}}, <_{KB})$ and copying c to it verbatim. If H is a solution to this instance, then by density of $(H, <_{KB})$, every pair of nodes in H has (at least) two incomparable extensions. One can then build a subcopy of $2^{<\mathbb{N}}$ inside H with a greedy algorithm which starts with any element of H as the root node and searches at each stage for two incomparable extensions of every node chosen so far.

The results of [DSV23] include, among many other things, significant strengthenings of some of the corollaries of Theorems 4.1 and 4.5 below, but otherwise do not overlap with ours, and are obtained by entirely different methods. However, the result of Theorem 4.1 has

$$\begin{array}{ccccccc}
\mathsf{iShuffle}_2 & \longleftarrow & \mathsf{iShuffle}_3 & \longleftarrow \cdots \longleftarrow & \mathsf{iShuffle} & \longleftarrow & \mathsf{iShuffle}_{\mathbb{N}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{Ind}\,\mathbb{Q}_2 & \longleftarrow & \mathsf{Ind}\,\mathbb{Q}_3 & \longleftarrow \cdots \longleftarrow & \mathsf{Ind}\,\mathbb{Q} & \longleftarrow & \mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{cRT}^1_2 & \longleftarrow & \mathsf{cRT}^1_3 & \longleftarrow \cdots \longleftarrow & \mathsf{cRT}^1_+ & \longleftarrow & \mathsf{cRT}^1_{\mathbb{N}}
\end{array}$$

Figure 1: Reductions between versions of cRT^1, $\mathsf{Ind}\,\mathbb{Q}$, and $\mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}}$. An arrow from Q to P signifies that $P <_{\mathrm{W}} Q$. No other reductions hold other than those implied by transitivity.

independently been superseded by Pauly [Pau24], who showed that $\mathsf{C}_{k+1} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$ where C_k outputs an element of a nonempty subset of $k = \{0, \ldots, k-1\}$ given an enumeration of its complement. His paper also improves on our Theorem 4.5 below, recovering it as part of a more general investigation. We present our proofs anyway in hope of the ideas therein being useful elsewhere, perhaps in non-reductions involving other indivisibility problems.

4.2. **The Weihrauch degree of $\mathsf{Ind}\,\mathbb{Q}$.** The main result of this section is that the reduction of $\mathsf{Ind}\,\mathbb{Q}_k$ to $\mathsf{iShuffle}_k$ is strict (Corollary 4.2), which follows from Theorem 4.1 below as $\mathsf{C}_{\mathbb{N}}$ is Weihrauch reducible to $\mathsf{TC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{iShuffle}_2$. The relationships between $\mathsf{Ind}\,\mathbb{Q}_k$, $\mathsf{TC}^k_{\mathbb{N}}$, and cRT^1_k are summarized in Figure 1. The fact that $\mathsf{iShuffle}_2 \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}}$ is another consequence of the theorem (Corollary 4.3), and we have $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \mathsf{cRT}^1_{\mathbb{N}}$ by Corollary 4.7 of Theorem 4.5 below. All other (non)reductions in the figure were justified in the previous section.

We will identify without comment natural numbers with the rationals they encode via a fixed computable presentation of $\mathbb{Q}$.

Theorem 4.1. $\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$.

Proof. It is known that $\mathsf{C}_{\mathbb{N}}$ is a fractal, so if $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q} = \bigsqcup_k \mathsf{Ind}\,\mathbb{Q}_k$ then $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$ for some k. It suffices to show this cannot occur. Fix k and suppose for the sake of contradiction that $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$ via Δ and Ψ.

Informally, the strategy is to wait for Ψ to output a number using some finite monochromatic set F as an oracle, and then try to diagonalize by enumerating that number. If we can extend from there to a $\mathsf{C}_{\mathbb{N}}$-instance for which F is still part of a valid $\mathsf{Ind}\,\mathbb{Q}_k$-solution, we are done. If not, then no matter how we extend, there will be an F-interval—an interval in the partition of $\mathbb{Q}$ induced by F—in which red points are scattered, if F was red. The particular F-interval where this is the case depends on the $\mathsf{C}_{\mathbb{N}}$-instance, so we account for all possibilities, repeating the same procedure as before in every F-interval, now looking for Ψ to converge on a blue set in each and diagonalizing again by enumerating all of its outputs into our instance of $\mathsf{C}_{\mathbb{N}}$. This completes the proof if $k = 2$ since blue points must be dense in any interval where red points are scattered. If there are more than two colors, and if again none of the blue sets found can possibly form part of a valid $\mathsf{Ind}\,\mathbb{Q}_k$-solution no matter how we continue to extend, then there will always be some subinterval induced by the blue sets in which blue points are scattered, and in particular there will be one in which *both* red and blue points are scattered. This is illustrated in Figure 2. Since every interval has a subinterval in which some color is dense, if the procedure is iterated, then eventually Δ cannot force any more colors to be scattered and so the final diagonalization attempt succeeds.

Now we proceed to the formal proof. We regard Ψ as computing a $\mathsf{C}_{\mathbb{N}}$-solution n if $\Psi^{f\oplus H}(0)\downarrow = n$, for an instance f of $\mathsf{C}_{\mathbb{N}}$ and $\mathsf{Ind}\,\mathbb{Q}_k$-solution H of Δ^f. A solution n is valid only if $n \notin \operatorname{ran} f$. Begin by finding a string $\sigma_0 \in \mathbb{N}^{<\mathbb{N}}$ and a finite set $S_\lambda \subset \mathbb{Q}$ such that $\Delta^{\sigma_0}(S_\lambda)\downarrow = 0$ (without loss of generality), and such that $\Psi^{\sigma_0 \oplus S_\lambda}(0)\downarrow = m_\lambda$ for some m_λ. Here λ denotes the empty string. One can obtain σ_0 and S_λ as initial segments of any $f \in \operatorname{dom} \mathsf{C}_{\mathbb{N}}$ and any H solving Δ^f. If $m_\lambda \in \operatorname{ran} \sigma_0$, the reduction fails already, so assume otherwise and let $\sigma_0' = \sigma_0^\frown m_\lambda$ be our first (or to be consistent later, zeroth) attempt at diagonalizing.

Suppose first there is a $g \in [\sigma_0'] \cap \operatorname{dom} \mathsf{C}_{\mathbb{N}}$ such that S_λ is extendible to a solution K of Δ^g. If so, then $\Psi^{g\oplus K}(0)\downarrow = m_\lambda$ but $m_\lambda \in \operatorname{ran} g$, defeating the reduction in this case. If not, write the elements of S_λ in increasing order as $x_1^\lambda < \ldots < x_{\ell(\lambda)}^\lambda$ where $\ell(\lambda) = |S_\lambda|$; let $x_0^\lambda = -\infty$ and $x_{\ell(\lambda)+1}^\lambda = +\infty$; let $X_\lambda = S_\lambda \cup \{x_0^\lambda, x_{\ell(\lambda)+1}^\lambda\}$; and let I_i^λ be the interval $(x_i^\lambda, x_{i+1}^\lambda)$. If S_λ is not extendible to a solution of Δ^g for any $g \in [\sigma_0'] \cap \operatorname{dom} \mathsf{C}_{\mathbb{N}}$, then for each g there is at least one $i \geq 0$ such that the set of color-0 points in I_i^λ with respect to Δ^g is scattered. Then the complementary set of points of Δ^g-colors $1, 2, \ldots, k-1$ is dense in that interval. We are then guaranteed to find monochromatic sets of at least one color $1, 2, \ldots, k-1$ in each such interval which witness convergence of Ψ on some extension of σ_0', and can repeat the diagonalization strategy on each, using a procedure described in the next paragraph. This will produce a tree structure of nested intervals labeled by finite strings β as displayed in Figure 2.

Let $L > 0$ and suppose σ_{L-1}' has already been found, and that the sets X_β and intervals I_i^β have been defined for all $|\beta| = L-1$ and $0 \leq i \leq \ell(\beta)$. At this stage, σ_{L-1}' represents the $(L-1)$st attempt—starting with the 0th—to diagonalize against the reduction, having already enumerated the outputs of Ψ on each of the sets S_δ for all $|\delta| \leq L-1$ for which such a set exists. Each S_δ has color $|\delta|$ and was found inside the interval I_i^γ where $\delta = \gamma^\frown i$, or simply inside $(-\infty, \infty)$ if $\delta = \lambda$ (corresponding to the "base case" $L = 0$ which was described in the last paragraph). We claim that there is a $\sigma_L \in [\sigma_{L-1}']$ which, intuitively speaking, sees Ψ converge on a set of color L in as many intervals I_i^β as possible, meaning no extension of σ_L does so in any additional interval. To be precise, for all $\rho \in [\sigma_L]$, if any $i \geq 0$ and β with $|\beta| = L-1$ are such that there exists a finite set $S \subset I_i^\beta$ which has Δ^ρ-color L and such that $\Psi^{\rho\oplus S}(0)\downarrow$, then the same is true of σ_L in place of ρ, for the same i (and possibly a different S). To prove the claim, consider the function $a\colon [\sigma_{L-1}'] \to \mathbb{N}$ which sends a string τ to the number of distinct intervals I_i^β, for $|\beta| = L-1$ and $0 \leq i \leq \ell(\beta)$, such that there exists $S_{\beta^\frown i} \subset I_i^\beta$ witnessing $\Delta^\tau(S_{\beta^\frown i})\downarrow = L$ and $\Psi^{\tau \oplus S_{\beta^\frown i}}(0)\downarrow$. Since there are only finitely many intervals I_i^β at hand, a is bounded, and any $\sigma_L \in [\sigma_{L-1}']$ witnessing its maximum value has the stated property.

Continue by picking any such σ_L, and fix a choice of $S_\alpha \subset I_i^\beta$ as above for any $\alpha = \beta^\frown i$ for which a set S_α could be found. Extend σ_L to σ_L' by enumerating the output m_α of $\Psi^{\sigma_L \oplus S_\alpha}(0)$ for each such α. Write $S_\alpha = \{x_1^\alpha < x_2^\alpha < \cdots < x_{\ell(\alpha)}^\alpha\}$, let $x_0^\alpha = x_i^\beta < x_1^\alpha$ and $x_{\ell(\alpha)+1}^\alpha = x_{i+1}^\beta > x_{\ell(\alpha)}^\alpha$, and let $X_\alpha = S_\alpha \cup \{x_0^\alpha, x_{\ell(\alpha)+1}^\alpha\}$. If for some $\alpha = \beta^\frown i$ of length L no S_α could be found, simply let $\ell(\alpha) = 0$ and $X_\alpha = \{x_0^\alpha, x_1^\alpha\} = \{x_i^\beta, x_{i+1}^\beta\}$. Then let $I_i^\alpha = (x_i^\alpha, x_{i+1}^\alpha)$ for all α of length L and all $i = 0, \ldots, \ell(\alpha)$. Thus if no set S_α as above could

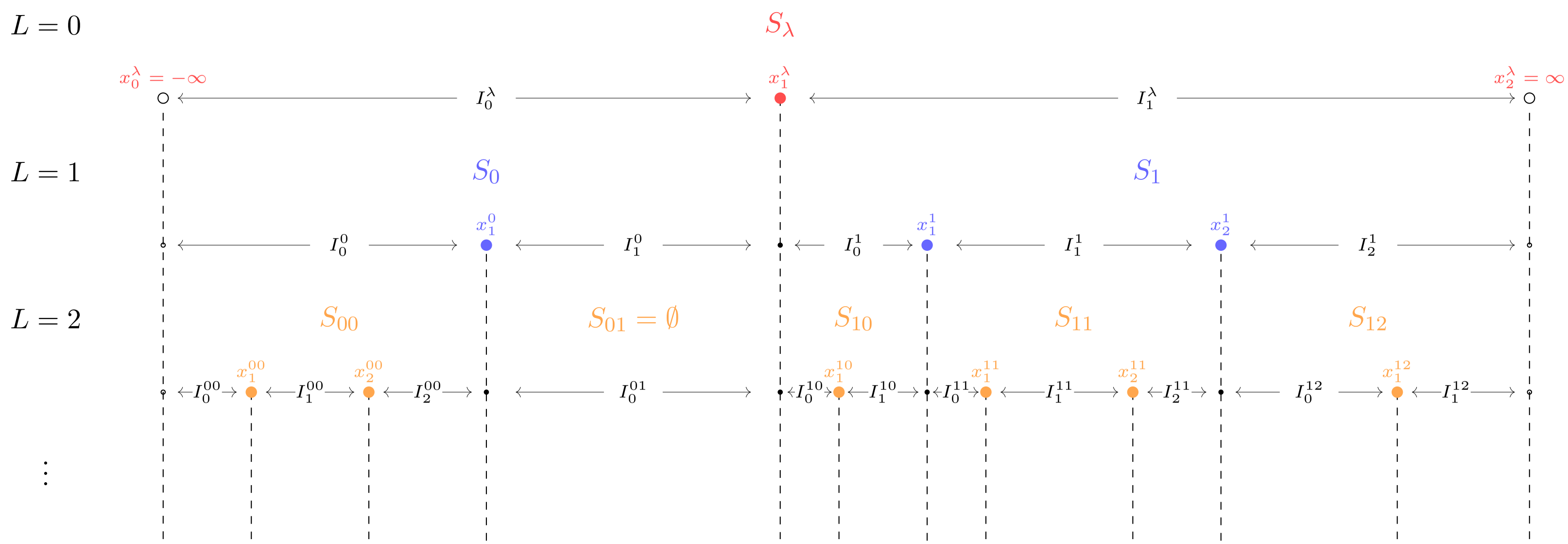

Figure 2: An illustration of an example diagonalization, with rational points vertically displaced to show which are added at each stage L. Stages are labeled by colors, with $L = 0$ being red, $L = 1$ blue, and $L = 2$ orange here. The point x_1^λ is carried through to become the largest element of X_0 and X_{01} and the smallest element of X_1 and X_{10}. One carries through x_1^0, x_1^1, and x_2^1 in a similar fashion, as well as $x_0^\lambda = -\infty$ and $x_2^\lambda = +\infty$. The latter two points are formally added merely to avoid having to treat the outermost intervals as special cases. In this example, no orange subset of I_1^0 could be found witnessing convergence of Ψ on any extension of σ_1, so we set $X_{01} = \{x_1^0, x_1^\lambda\} = \{x_1^0, x_2^0\}$. At the end of the procedure, at least one of I_0^λ, I_1^λ will have red points scattered; at least one of I_0^0, I_1^0 and one of I_0^1, I_1^1, I_2^1 will have blue points scattered; I_0^{01} will have orange points scattered, as will at least one of the I_i^{00}s, one of the I_i^{10}s, one of the I_i^{11}s, and one of the I_i^{12}s; and so on.

be found for $\alpha = \beta^\frown i$, then we have $I_0^\alpha = I_i^\beta$, so at the next stage one searches for an $S_{\alpha^\frown 0}$ of color $L+1$ in the same interval rather than splitting it further.

If there is a $g \in [\sigma'_L] \cap \operatorname{dom} \mathsf{C}_\mathbb{N}$ and an α of length L such that S_α is extendible to a color-L solution of Δ^g, then this g witnesses the failure of the reduction since for some solution $U \supset S_\alpha$ of Δ^g, we have $\Psi^{g \oplus U}(0)\downarrow = m_\alpha$ while $m_\alpha \in \operatorname{ran} g$. Suppose, then, that no such g exists. This means that for all $g \in [\sigma'_L] \cap \operatorname{dom} \mathsf{C}_\mathbb{N}$, for all β with $|\beta| = L-1$, there is an $i \geq 0$ such that the set of color-L points in I_i^β is scattered with respect to Δ^g. If $L < k-1$, then continue inductively as above by finding sets S_ξ for $|\xi| = L+1$ and a string $\sigma_{L+1} \supset \sigma'_L$. But if $L = k-1$, then we claim that in fact for any $g \in [\sigma'_L] \cap \operatorname{dom} \mathsf{C}_\mathbb{N}$, there must be an S_α with $|\alpha| = L$ that is extendible to a color-$(k-1)$ solution of Δ^g. To see why, recall that by assumption at this stage, for any such g, in particular there is an i_0 such that color-0 points are scattered in $I_{i_0}^\lambda$. In turn, there is an i_1 such that color-1 points are also scattered in $I_{i_1}^{i_0}$; an i_2 such that color-2 points are also scattered in $I_{i_2}^{i_0 i_1}$; and so on, so that there is ultimately an i_{k-2} such that the points of colors $0, 1, \ldots, k-2$ are all scattered in the interval $I_{i_{k-2}}^\beta$, where $\beta = i_0 i_1 \cdots i_{k-3}$ (or $\beta = \lambda$ if $k = 2$). It follows that the set of points of color $k-1$ is dense in $I_{i_{k-2}}^\beta$. In particular, the set of color-$(k-1)$ points in $I_{i_{k-2}}^\beta$ is a solution of Δ^g, so if $\alpha = i_0 i_1 \ldots i_{k-2}$, then during the procedure we must have been able to find a set S_α of color $k-1$ with $\Psi^{\sigma_L \oplus S_\alpha}(0)$ converging. Hence for this g and α, there is a solution $U \supset S_\alpha$ of Δ^g with $\Psi^{g \oplus U}(0)\downarrow = m_\alpha$ while $m_\alpha \in \operatorname{ran} g$, defeating the reduction and completing the proof. □

Corollary 4.2. *For all $k \geq 2$, the Weihrauch reducibility of* $\mathsf{Ind}\, \mathbb{Q}_k$ *to* $\mathsf{iShuffle}_k$ *is strict.* □

Corollary 4.3. $\mathsf{iShuffle}_2 \not\leq_\mathrm{W} \mathsf{Ind}\, \mathbb{Q}_\mathbb{N}$*, and hence in particular* $\mathsf{iShuffle}_\mathbb{N} \not\leq_\mathrm{W} \mathsf{Ind}\, \mathbb{Q}_\mathbb{N}$.

Proof. Recall that $\mathsf{iShuffle}_2 \equiv_\mathrm{W} \mathsf{TC}_\mathbb{N}$. The latter problem is known to be a total fractal. We have by Proposition 3.3 that $\mathsf{Ind}\, \mathbb{Q}_\mathbb{N} \equiv_\mathrm{W} \mathsf{Ind}\, \mathbb{Q} * \mathsf{C}_\mathbb{N}$, and it follows that if $\mathsf{TC}_\mathbb{N} \leq_\mathrm{W} \mathsf{Ind}\, \mathbb{Q}_\mathbb{N}$ then $\mathsf{TC}_\mathbb{N} \leq_\mathrm{W} \mathsf{Ind}\, \mathbb{Q}$, contradicting Theorem 4.1. □

Theorem 4.1 and its corollaries show that $\mathsf{Ind}\, \mathbb{Q}$ is uniformly rather weak, but in a sense it is not too easy to solve either, in that it cannot be solved by any problem having a c.e. approximation in the following sense. Let Φ be a universal functional.

Definition 4.4. We say a problem P is *pointwise c.e. traceable*, or p.c.e.t. for short, if there is an index i such that for all $p \in \operatorname{dom} P$, $\Phi_{\langle i,p\rangle}(0)$ converges and outputs some e such that W_e^p is finite and contains at least one index j with $\Phi_{\langle j,p\rangle} \in P(p)$.

Intuitively, this means there is a uniform procedure to enumerate a finite list of potential solutions to any instance of P, at least one of which will turn out to be correct. Examples of pointwise c.e. traceable problems include $\mathsf{RT}^1_\mathbb{N}$ and any problem with finite computable codomain.

Theorem 4.5. *If P is any pointwise c.e. traceable problem, then* $\mathsf{Ind}\, \mathbb{Q}_2 \not\leq_\mathrm{W} P$.

Hence none of $\mathsf{Ind}\, \mathbb{Q}_k$, $\mathsf{Ind}\, \mathbb{Q}$, and $\mathsf{Ind}\, \mathbb{Q}_\mathbb{N}$ are reducible to P either. Although this result was found independently in the course of the present work, the core idea of its proof was outlined in [FP17, §5], where the authors call it a "disjoint extension commitment" of ER^1 (or $\mathsf{Ind}\, \mathbb{Q}$), and may have appeared elsewhere. The idea is that if $\mathsf{Ind}\, \mathbb{Q}_2 \leq_\mathrm{W} P$, then when the return functional Ψ outputs any points $x < y < z$, it commits to outputting monochromatic densely ordered sets both in (x, y) and in (y, z). If there are only finitely many possible

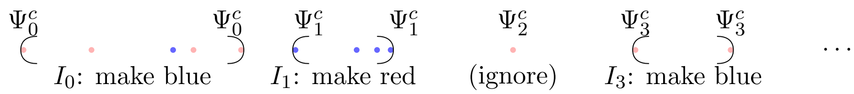

Figure 3: An example diagonalization against functionals $\Psi_0^c, \Psi_1^c, \dots$. The points shown have already been colored red or blue by c at this stage, and enumerated by Ψ_i^c for some $i \in N(s)$. Ψ_0^c, Ψ_1^c, and Ψ_3^c have enumerated enough points so that we can find three disjoint intervals I_0, I_1, I_3 whose endpoints are respectively among the domains of $\Psi_0^c, \Psi_1^c, \Psi_3^c$ up to this point. Since the endpoints of I_0 are red, we plan to make all future points in I_0 blue, and similarly for I_1 and I_3. At this stage, Ψ_2^c has not enumerated enough points to find an I_2 disjoint from $I_0 \cup I_1 \cup I_3$, so we ignore it for now.

solutions that need be considered as oracles for Ψ, then once Ψ has output enough points per solution, we can choose a finite set of pairwise disjoint intervals, one per solution, and diagonalize separately in each by filling it in with the opposite color as its endpoints. This works as long as we know the number of solutions of the computed P-instance will not grow without bound—this is what the p.c.e.t. condition is used for.

Proof of Theorem 4.5. Suppose $\mathsf{Ind}\,\mathbb{Q}_2 \leq_{\mathrm{W}} P$ via Δ and Ψ. We reach a contradiction by building a coloring $c = \lim_s c_s$ of $\mathbb{Q}$ by finite extension to witness the failure of this reduction, beginning with $c(0) = 0$.

Let i be an index associated to P as in Definition 4.4, and let $N(s)$ be either $W_x^{\Delta(c)}[s]$ if $\Phi_{\langle i,\Delta(c)\rangle}(0)[s]\downarrow= x$, or $\emptyset$ if this computation diverges. For each $j \in N(s)$, let $\Psi_j^c = \Psi(\Phi_{\langle j,\Delta(c)\rangle} \oplus c)$, the computation of the return functional using the jth $\Delta(c)$-solution furnished by Φ as an oracle, along with the original coloring c. Since there is always some $j \in \bigcup_s N(s)$ such that $\Phi_{\langle j,\Delta(c)\rangle}$ solves $\Delta(c)$, at least one Ψ_j^c must be a solution to c in the end, so it is enough to diagonalize against every Ψ_j^c without regard to its eventual correctness.

Define a recurrence relation $r(n)$ by letting $r(1) = 3$ and $r(n+1) = 2nr(n) + 2$ for $n \geq 1$.

Lemma 4.6. *If sets $S_0, \dots, S_{k-1}$ each contain at least $r(k)$ points, then there are pairwise disjoint rational intervals $I_0, \dots, I_{k-1}$ such that the endpoints of I_i are in S_i for each i.*

Proof. By induction on k. The case $k = 1$ is immediate, so suppose the claim holds for all $i \leq k$ and that we have $k + 1 > 1$ sets $S_0, \dots, S_{k-1}, S_k$ each with at least $r(k+1)$ points. For purposes of the inductive step, we discard all but $2r(k)$ points from the first k sets $S_0, \dots, S_{k-1}$. Then the $2kr(k)$ total points in $S_0 \cup \dots \cup S_{k-1}$ divide $\mathbb{Q}$ into $2kr(k) + 1$ open intervals. If S_k has at least $2kr(k) + 2$ points, by the pigeonhole principle, at least one of those $2kr(k) + 1$ open intervals will contain two distinct points $a_k, b_k \in S_k$. We in particular choose a_k and b_k to be adjacent among the elements of S_k, then set $I_k = (a_k, b_k)$. This splits $\bigcup_{i<k} S_i$ into two groups, the points below a_k and those above b_k. Using the pigeonhole principle again, for each $i < k$, one of those groups contains at least $2r(k)/2 = r(k)$ elements of S_i. The inductive hypothesis can now be applied to the collection of (at most k) sets having at least $r(k)$ points below a_k, and separately to the collection of sets having at least $r(k)$ points above b_k, to produce a set of disjoint rational intervals as required. □

Now we proceed to the main argument, which is by finite injury (without a priority order). Let $A(s)$ be the set of $i \in N(s)$ such that $\Psi_i^c[s]$ has output at least one point. Let

$V(s)$ be the set of $i \in A(s)$ such that $\Psi_i^c[s]$ has output at least $r(|A(s)|)$ points. At stage $s+1$, for each $i \in V(s)$, if Ψ_i does not have a follower, assign it a follower which is a (code for a) rational interval I_i as furnished by Lemma 4.6, applied to the sets $S_i = \Psi_i^c[s]$ for $i \in V(s)$. Thus the endpoints of I_i were enumerated by Ψ_i^c by stage s, and $I_i \cap I_j = \emptyset$ if $i \neq j \in V(s)$. If there is any $j \in A(s+1) \setminus A(s)$, cancel the followers of all Ψ_i which have one at stage $s+1$. Then if $s+1$ is an element of I_i for some $i \in V(s)$ and a is an endpoint of I_i, color $c_{s+1}(s+1) = 1 - c_s(a)$. Otherwise, if $s+1$ is not an element of any interval I_i, color $c_{s+1}(s+1) = 0$. (See Figure 3 for an example of how this might look at a particular stage.)

Because $A(s)$ eventually stabilizes, each Ψ_i will have a follower canceled at most finitely many times, and so the set of intervals I_i also stabilizes. Any Ψ_i which outputs infinitely many points will have $i \in V(s)$ for large enough s, so for each such i, there are two points output by Ψ_i^c between which only finitely many are of the same color. Therefore Ψ_i^c does not compute a solution to c. □

Note that the construction above can be done computably.

Corollary 4.7.

(i) $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \mathsf{cRT}^n_{\mathbb{N}}$ *for all* n.
(ii) $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} (\mathsf{LPO}^{(n)})^*$ *for all* n.
(iii) $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \mathsf{cShuffle}$.

Proof.

(i) $\mathsf{cRT}^n_{\mathbb{N}}$ is p.c.e.t. since the set of solutions to c is contained in $\operatorname{ran} c$.
(ii) If there are k instances of $\mathsf{LPO}^{(n)}$ given in parallel, at most 2^k distinct solutions are possible, and they can each be encoded as a single natural number. So in particular $(\mathsf{LPO}^{(n)})^*$ is p.c.e.t.
(iii) This follows from (ii) as $\mathsf{cShuffle} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$ [PPS24]. □

Before continuing to the next section, we pause to make some observations about p.c.e.t. problems. Pointwise c.e. traceability can be viewed as a generalization of c.e. traceability, which is the special case where P is a single-valued function from $\mathbb{N}$ to $\mathbb{N}$ (see for instance [Soa16, §11.4] for the definition). It is also related to the notion of a pointwise finite problem, defined in [GPV21], which is a problem such that every instance has finitely many solutions. However, a p.c.e.t. problem is not necessarily pointwise finite, one counterexample being $\mathsf{C}_{\mathbb{N}}$. And the existence of functions $\mathbb{N} \to \mathbb{N}$ which are not c.e. traceable shows that even first-order pointwise finite problems may not be p.c.e.t.

The class of p.c.e.t. problems has some attractive algebraic properties:

- If P and Q are p.c.e.t., then so are $P \times Q$ and P^*.
- If P is p.c.e.t. and $Q \leq_{\mathrm{W}} P$, then Q is p.c.e.t.

(These properties hold for problems P and Q on any represented spaces, as is straightforward to show.) On the other hand,

- $P * Q$ may not be p.c.e.t. even if both P and Q are. For example, $\mathsf{C}_{\mathbb{N}}$ and LPO' are both p.c.e.t. but $\mathsf{C}_{\mathbb{N}} * \mathsf{LPO}' \geq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}} \geq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_2$ [BG21, Corollary 8.10].
- Neither P^{u*} nor $P^{\diamond}$ need be p.c.e.t. if P is. It follows from [SV23, Theorem 7.2] that $(\mathsf{LPO}')^{u*} \equiv_{\mathrm{W}} (\mathsf{LPO}')^{\diamond} \equiv_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}}$, and $\mathsf{C}'_{\mathbb{N}} >_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}$ [BG21, Corollary 8.14]. (See [SV23] for definitions of the undefined notation used here.)

Just as the fact that $\mathsf{LPO}' |_{\mathrm{W}} \lim$ (which is known in the literature, see e.g. [BG21]) shows that pointwise c.e. traceability is logically incomparable with limit computability, the

$$\begin{array}{ccccccccc}
\mathsf{RT}^2_2 & \longleftarrow & \mathsf{RT}^2_3 & \longleftarrow & \mathsf{RT}^2_4 & \longleftarrow \cdots \longleftarrow & \mathsf{RT}^2_+ & \longleftarrow & \mathsf{RT}^2_{\mathbb{N}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{Ind}\,\mathscr{E}_2 & \longleftarrow & \mathsf{Ind}\,\mathscr{E}_3 & \longleftarrow & \mathsf{Ind}\,\mathscr{E}_4 & \longleftarrow \cdots \longleftarrow & \mathsf{Ind}\,\mathscr{E} & \longleftarrow & \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{SRT}^2_2 & \longleftarrow & \mathsf{SRT}^2_3 & \longleftarrow & \mathsf{SRT}^2_4 & \longleftarrow \cdots \longleftarrow & \mathsf{SRT}^2_+ & \longleftarrow & \mathsf{SRT}^2_{\mathbb{N}}
\end{array}$$

Figure 4: Reductions between SRT^2_k, $\mathsf{Ind}\,\mathscr{E}_k$, and RT^2_k. An arrow from Q to P signifies that $P <_{\mathrm{W}} Q$. No other Weihrauch reductions hold other than those implied by transitivity. The diagram remains true if $<_{\mathrm{W}}$ is replaced by $<_{\mathrm{c}}$, except that the reductions $\mathsf{SRT}^2_k \leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_k$ are not known to be strict, and that the two rightmost entries in each row collapse under $\equiv_{\mathrm{sc}}$.

fact that LPO' is p.c.e.t. shows the concept to be strictly more general than computability with finitely many mind changes: the latter can be characterized by Weihrauch reducibility to $\mathsf{C}_{\mathbb{N}}$, which is p.c.e.t. and known to be Weihrauch incomparable with LPO'.

The study of p.c.e.t. problems has been continued by Pauly [Pau24], who calls them *eventually-finitely guessable* and develops the idea as part of a family of similar notions.

5. The countable equivalence relation

The countable equivalence relation $\mathscr{E}$ was defined in the introduction. We identify $E = \operatorname{dom}\mathscr{E}$ with $\mathbb{N}\times\mathbb{N}$, viewing (x, y) as the yth element of the xth equivalence class. We also refer to the xth equivalence class as the "xth column" of $\mathscr{E}$.

The proof of this theorem was obtained jointly with Linda Westrick.

Theorem 5.1. *For all $k \geq 2$, $\mathsf{SRT}^2_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_k \leq_{\mathrm{W}} \mathsf{RT}^2_k$, but $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ and $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$.*

Since $\mathsf{SRT}^2_j \not\leq_{\mathrm{c}} \mathsf{RT}^2_k$ whenever $j > k \geq 2$ [HJ16, Pat16], we obtain Figure 4, a very similar diagram to that shown for $\mathsf{Ind}\,\mathbb{Q}$ in Figure 1. To see why $\mathsf{SRT}^2_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^2_+$, notice that $\mathsf{SRT}^2_{\mathbb{N}}$ is a fractal; the proof is along the same lines as those of Proposition 3.5(ii) which implies the same for $\mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$. Then if there were such a reduction, we would have $\mathsf{SRT}^2_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}^2_k$ for some k, and in particular $\mathsf{SRT}^2_{k+1} \leq_{\mathrm{W}} \mathsf{RT}^2_k$, a contradiction. The remaining nonreductions immediately follow.

The rest of the section is occupied with the proof of Theorem 5.1, which is broken into four lemmas. The first two establish that $\mathsf{SRT}^2_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_k \leq_{\mathrm{W}} \mathsf{RT}^2_k$, the third shows $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ and thus gives a separation between $\mathsf{Ind}\,\mathscr{E}_k$ and RT^2_k, and the fourth shows $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$ and so separates $\mathsf{Ind}\,\mathscr{E}_k$ from SRT^2_k.

We use the notation $\{x, y\}$ for unordered pairs, and omit the extra parentheses from $c(\{x, y\})$ and $c((x, y))$ to reduce visual clutter.

Lemma 5.2. $\mathsf{Ind}\,\mathscr{E}_k \leq_{\mathrm{W}} \mathsf{RT}^2_k$.

Proof. Let $c\colon \mathbb{N} \to k$ be a coloring of E. Define an RT^2_k-instance $d\colon [\mathbb{N}]^2 \to k$ by

$$d\{x, y\} = c(x, y) \quad \text{if } x < y.$$

Let $\tilde{H}$ be an infinite homogeneous set for d, and let H be the $\tilde{H}$-computable set

$$H = \{ (x, y) : x \leq y \in \tilde{H} \} \subseteq \mathbb{N}^2.$$

It is clear that H is monochromatic for c. For each $x \in \tilde{H}$, $(x, y) \in H$ for all of the infinitely many $y > x$ in $\tilde{H}$, so H induces a substructure isomorphic to $\mathscr{E}$. □

Lemma 5.3. $\mathsf{SRT}^2_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_k$.

Proof. Let $c\colon [\mathbb{N}]^2 \to k$ be an instance of SRT^2_k. Define $d\colon E \to k$ by

$$d(x, y) = \begin{cases} c\{x, y\}, & \text{if } x < y \\ 0, & \text{otherwise.} \end{cases}$$

Let H be an $\mathsf{Ind}\,\mathscr{E}_k$-solution for d. We build an SRT^2_k-solution for c as follows: start with any $(x_1, y) \in H$ with $x_1 < y$. Then find another column $x_2 > x_1$ represented in H such that $c\{x_1, x_2\} = c\{x_1, y\}$. There is such an x_2 by stability of the coloring c: the d-color of H must be the same as the stable c-color of column x_1, so for all large enough x, $c\{x_1, x\} = c\{x_1, y\} = d(x_1, y)$. Since there are infinitely many equivalence classes of E represented in H, and the equivalence class of x_1 has infinitely many points of this color, there is eventually a column n represented in H with $d(x_1, x) = d(x_1, y)$, and we can take x_2 to be this x. (Note (x_1, x_2) need not be in H.) Next, find a column $x_3 > x_2$ represented in H such that

$$c\{x_1, x_2\} = c\{x_1, x_3\} = c\{x_2, x_3\}.$$

Again, this must be possible by stability and the existence of infinitely many different columns represented in H. Proceed in the same way, for each i finding a column x_i of H with $x_i > x_{i-1}$ such that $c\{x_j, x_k\} = c\{x_1, x_2\}$ for all $1 \leq j < k \leq i$. The set $\{ x_i : i \in \mathbb{N} \}$ is then an infinite homogeneous set for c. □

It is clear that the proofs of the above lemmas do not depend on k and thus extend to show $\mathsf{SRT}^2_+ \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E} \leq_{\mathrm{W}} \mathsf{RT}^2_+$ and $\mathsf{SRT}^2_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$.

Corollary 5.4. *For all $k \geq 2$, $\mathsf{Ind}\,\mathscr{E}_{k+1} \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_k$ and $\mathsf{Ind}\,\mathscr{E} \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_k$.*

Proof. Both statements follow from the above lemmas together with the fact that $\mathsf{SRT}^2_{k+1} \not\leq_{\mathrm{c}} \mathsf{RT}^2_k$ [Pat16, Corollary 3.6]. □

Lemma 5.3 also enables us to recover the following result originally due to Ackerman, Freer, Reimann, and Westrick [AFRW19]:

Corollary 5.5. *$\mathscr{E}$ is not computably indivisible.*

Proof. By a result of Jockusch (see [Hir15, Exercise 6.35]), there is a computable instance of SRT^2_2 with no computable solution. The same must be true for $\mathsf{Ind}\,\mathscr{E}_2$ by Lemma 5.3. □

Lemma 5.6. $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$.

Proof. We show that $\mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ admits Δ^0_2 solutions, i.e., every computable instance has a Δ^0_2 solution. As RT^2_2 does not share this property [Joc72, Corollary 3.2], the separation follows.

Let c be a computable coloring of E with $\operatorname{ran} c \subseteq k$ for some k. Suppose c stabilizes in infinitely many columns. Nonuniformly pick a color i which is the stable color of infinitely many columns, and use $\emptyset'$ to enumerate the set S of such columns by asking, for each x, y, if there is $z > y$ with $c(x, z) \neq i$, and putting $x \in S$ if the answer is 'no'. As a relativization of the fact that every infinite c.e. set has an infinite computable subset, there is an infinite Δ^0_2 subset of S, and the set of points of color i in all such columns is a Δ^0_2 solution to c.

If instead c stabilizes in only finitely many columns, nonuniformly delete those columns; the remaining columns each have infinitely many points of at least two different colors. Suppose there are colors $i_1 \neq i_2$ such that infinitely many columns include only finitely many points of colors other than i_1 or i_2. The set S of such columns can again be enumerated by $\emptyset'$ by asking, for each x, y, if there is $z > y$ with $c(x, z) \notin \{i_1, i_2\}$, and putting $x \in S$ if the answer is 'no'. S has an infinite Δ^0_2 subset, and the set of points of color i_1 (or of i_2) in the columns in that subset is a Δ^0_2 solution to c because, by assumption, all columns of S have infinitely many points both of color i_1 and of color i_2.

In general, let $n \geq 1$ be the least possible number of distinct colors $i_1, \ldots, i_n$ such that there is an infinite set S of columns each of which includes only finitely many points of colors other than $i_1, \ldots, i_n$. That is, n is such that only finitely many columns have infinitely many points of $n-1$ or fewer different colors. Nonuniformly delete this finite set of columns, so that every column remaining has infinitely many points of each of at least n colors, and fix a particular choice of $i_1, \ldots, i_n$ as above. Then for this set of colors $\{i_1, \ldots, i_n\}$, $\emptyset'$ can enumerate S by letting $x \in S$ if we find a y such that there is no $z > y$ having $c(x, z) \notin \{i_1, \ldots, i_n\}$. There is an infinite Δ^0_2 subset of S, and the set of points of color i_j in the columns of this subset, for any $1 \leq j \leq n$, is a Δ^0_2 solution of c. □

Lemma 5.7. $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$.

Proof. $\mathsf{Ind}\,\mathscr{E}_2$ is a total fractal and $\mathsf{SRT}^2_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{SRT}^2_{+} * \mathsf{C}_{\mathbb{N}}$ (see Proposition 3.5 and the discussion after Proposition 3.3), so if $\mathsf{Ind}\,\mathscr{E}_2 \leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$, then there is some k with $\mathsf{Ind}\,\mathscr{E}_2 \leq_{\mathrm{W}} \mathsf{SRT}^2_k$. Suppose there is such a reduction via the functionals Δ and Ψ. We use lowercase Latin letters for instances of $\mathsf{Ind}\,\mathscr{E}_2$ and lowercase Greek letters for their initial segments.

We will find (noncomputably) a finite sequence of strings $\sigma_0 \subset \sigma_1 \subset \cdots$ such that for some s, there is an $\mathsf{Ind}\,\mathscr{E}_2$-instance $c \in [\sigma_s]$ which defeats the Weihrauch reduction. To accomplish this, we exploit the requirement for Δ to produce a stable coloring in order to restrict our attention to sets of columns H which are guaranteed to extend to an SRT^2_k-solution as long as there is any such solution of the same color as $[H]^2$, no matter how we continue to extend σ_s. Then we use these sets as oracles to diagonalize against Δ and Ψ by adding finitely many locks on the columns of E (to be explained below), represented by a lock function $L = \bigcup_s L_s \colon \subseteq \mathbb{N} \to 2$ that will be updated finitely many times. These locks are in fact the same as those used in Cohen forcing with locks, but we do not formally use a notion of forcing here and the argument does not necessarily produce a generic c. After we lock a column, either Ψ will be forced to output a non-monochromatic set, or Δ will be forced to prevent H from being an initial segment of any solution—and so will prevent there from being *any* solution of H's color. This can be carried out for every color used by Δ, so eventually the diagonalization will succeed.

For any string σ and lock function L, let $[\sigma, L]$ be the set of (finite or infinite) $\tau \in [\sigma]$ such that whenever $L(x) \in \{0, 1\}$ is defined, if (x, y) is newly colored by τ, then $\tau(x, y) = L(x)$. "Newly colored" simply means that τ is defined on (x, y) while σ is not. So $[\sigma, L]$ is the set of extensions of σ which respect the lock function L by only adding points of color $L(x)$ in column x. Observe that $[\sigma, L'] \subset [\sigma, L]$ if $L' \supset L$.

The central claim that makes this proof work is as follows: for any σ, any SRT^2_k-column n, and any lock function L,

$$\exists\tau\in[\sigma,L]\ \exists i\in\mathbb{N}\ \forall\rho\in[\tau,L]\ (\text{if } \Delta^\rho \text{ newly colors } \{n,m\} \text{ where } m>n, \text{ then } \Delta^\rho\{n,m\}=i).$$

That is, we can pass to an extension of σ such that among all further extensions respecting the lock function, Δ will never change its mind again about the stable color of column n, and in fact will never again add a point of any other color to it. We say τ L-forces column n to be color i when this happens. If the claim were false for some σ, n, and L, then for every $\tau\in[\sigma,L]$ and $i\in k$, there would be a further $\rho(\tau)\in[\tau,L]$ such that $\Delta^{\rho(\tau)}$ adds a point not of color i into column n. Letting $\tau_0=\sigma$ and $\tau_{s+1}=\rho(\tau_s)$ for all s, the $\mathsf{Ind}\,\mathscr{E}_2$-instance $d=\lim_s\tau_s$ is such that Δ^d has infinitely many points of at least two different colors in column n, so is not an SRT^2_k-instance, a contradiction. A key point is that this argument works for any L independently of σ, so that even if we change L partway through the proof we can always extend any σ_s to a σ_{s+1} which L-forces a fresh column to be some color. A column already L-forced remains L'-forced to be the same color for any $L'\supset L$.

Now we describe the procedure to find (σ_s) and (L_s) in detail. Let $D\subseteq k$ be the set of "already-diagonalized" colors, starting with $D=\emptyset$; we will use D to keep track of which colors have already witnessed convergence of Ψ and triggered an update of the lock function. As long as $D\subsetneq k$, the colors in D at stage s will be exactly those of which no Δ^c-solution can be found for any $c\in[\sigma_s,L_s]$. Let $L_0=\emptyset$ and let σ_0 be a string L_0-forcing column 0 to be some color. At stage $s>0$, let $\sigma_s\in[\sigma_{s-1},L_{s-1}]$ be either a string which L_{s-1}-forces column s to be some color if it had not yet been so forced, or an arbitrary string otherwise. For each $i\notin D$, search for a pair (x,y) already colored by σ_s with $x\notin\operatorname{dom}L_{s-1}$ (i.e., with column x not already locked), together with a finite set H such that $[H]^2$ is Δ^{σ_s}-monochromatic with color i, such that σ_s L_{s-1}-forces every column in H to be color i, and such that $\Psi^{\sigma_s\oplus H}(x,y)\downarrow=1$. Such x, y, and H must eventually be found for some i and s if Ψ is to compute an $\mathsf{Ind}\,\mathscr{E}_2$-solution, and we can take $x\notin\operatorname{dom}L_{s-1}$ because Ψ will output elements of infinitely many columns of E whereas $\operatorname{dom}L_{s-1}$ is finite. That we can also take $i\notin D$ is justified below. If x and y are found at stage s, diagonalize by setting $L_s(x)=1-\sigma_s(x,y)$; that is, we lock column x to be the opposite color that Ψ already committed to there. End stage s by putting $i\in D$. If instead no such x, y, or H are found for any i, let $L_s=L_{s-1}$ and end the stage.

If H is as in the previous paragraph, then H is a valid initial segment of some solution to Δ^d for *any* $d\in[\sigma_s,L_{s-1}]$ for which infinitely many SRT^2_k-columns stabilize to i, and in particular for any such $d\in[\sigma_s,L_s]$. This is because starting from any Δ^d-solution G, one can truncate G to $\hat{G}$ by selecting only the columns whose indices are higher than the point at which any of the columns in H stabilize, producing an infinite $\hat{G}$ with $[H\cup\hat{G}]^2$ monochromatic. Hence either H extends to a solution of color i (for some d as above), or there is no solution of color i (for any d as above). In the first case, d would witness the failure of the Weihrauch reduction, because if H extends to a Δ^d-solution K, then $\Psi^{d\oplus K}$ outputs a point in E whose color is shared by only finitely many other points in the same column. If such a d exists then the procedure ends at this stage. In particular, this must happen if $D=k$, because then some H, x, and y were already found for every $i\in k$, and for any $d\in[\sigma_s,L_s]$, at least one of these Hs must extend to a valid solution of Δ^d.

Otherwise, for this i and indeed for all $i\in D$, there is no $d\in[\sigma_s,L_s]$ with Δ^d having a solution of color i. This means that only finitely many columns of any such Δ^d stabilize to a color in D, and it must still be possible to L_s-force infinitely many columns to be a color not in D. Hence we can continue the procedure to extend σ_s and search for a suitable H of some color $i \notin D$. This completes the proof, since the procedure will eventually diagonalize against every color of which it is still possible for there to be an SRT^2_k-solution. □

6. Further directions

Solutions of $\mathsf{Ind}\,\mathbb{Q}$ are rather nebulous: for example, if x and y are two elements of a solution H, one can delete the whole interval $[x, y]$ from H and still obtain a solution. The seeming weakness of $\mathsf{Ind}\,\mathbb{Q}$ may be a consequence of this nebulosity, and it is unclear how much power it derives from the fact that it outputs a second-order object, i.e., a set of rationals. An investigation of these properties can be put on precise footing with the following notions introduced in [DSY23] and [GPV21], respectively:

- The *first-order part* of a problem P, ${}^1P$, is the strongest first-order problem Weihrauch reducible to P.
- The *deterministic part* of P, $\operatorname{Det} P$, is the strongest problem Weihrauch reducible to P for which every instance has a unique solution.

Question 6.1. How strong exactly are ${}^1\,\mathsf{Ind}\,\mathbb{Q}$ and $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q})$?

The first-order part of $\mathsf{Ind}\,\mathbb{Q}$ has been studied in recent work by Dzhafarov, Solomon, and Valenti on the tree pigeonhole principle TT^1_+ [DSV23], which as mentioned earlier is Weihrauch equivalent to $\mathsf{Ind}\,\mathbb{Q}$. They showed that none of the problems TT^1_k, TT^1_+, or $\mathsf{TT}^1_\mathbb{N}$ are Weihrauch equivalent to any first-order problem, and indeed ${}^1\mathsf{TT}^1_\mathbb{N} \equiv_\mathrm{W} \mathsf{RT}^1_\mathbb{N}$. But ${}^1\mathsf{TT}^1_k$ turns out to be strictly stronger than RT^1_k for all $k \geq 2$; whether ${}^1\mathsf{TT}^1_k$ and ${}^1\mathsf{TT}^1_+$ have precise characterizations in terms of known Weihrauch degrees is open. Regarding $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q})$, Manlio Valenti has pointed out to the author that since $\operatorname{Det}(\mathsf{RT}^1_k) \equiv_\mathrm{W} \lim_k$ and $\operatorname{Det}(\mathsf{TC}^*_\mathbb{N}) \equiv_\mathrm{W} \mathsf{C}_\mathbb{N} \equiv_\mathrm{W} \lim_\mathbb{N}$, where $\lim_k$ maps an eventually constant element of $k^\mathbb{N}$ to its limit and similarly for $\lim_\mathbb{N}$, we must have $\lim_k <_\mathrm{W} \operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}) <_\mathrm{W} \lim_\mathbb{N}$ for all k. (The second reduction is strict by Theorem 4.1.) Precise characterizations of the degrees of $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}_k)$ and of $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q})$ have not been established. On the other hand, $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}_\mathbb{N}) \equiv_\mathrm{W} \mathsf{C}_\mathbb{N}$ since $\mathsf{C}_\mathbb{N} \leq_\mathrm{W} \mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$ and, using [GPV21, Theorem 3.9], $\operatorname{Det}(\mathsf{TC}^*_\mathbb{N} * \mathsf{C}_\mathbb{N}) \leq_\mathrm{W} \operatorname{Det}(\mathsf{TC}^*_\mathbb{N}) * \mathsf{C}_\mathbb{N} \equiv_\mathrm{W} \mathsf{C}_\mathbb{N} * \mathsf{C}_\mathbb{N} \equiv_\mathrm{W} \mathsf{C}_\mathbb{N}$.

One consequence of the above facts is that both the first-order and the deterministic parts of $\mathsf{Ind}\,\mathbb{Q}$ are p.c.e.t., and so one cannot replace $\mathsf{Ind}\,\mathbb{Q}_2$ with ${}^1\,\mathsf{Ind}\,\mathbb{Q}_2$ or $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}_2)$ in Theorem 4.5.

Turning to $\mathscr{E}$, Theorem 5.1 is intriguing given the great interest in RT^2_2 in reverse mathematics since the 1970s. Cholak, Jockusch, and Slaman in [CJS01] proved that RT^2_2 can be logically decomposed into the conjunction of SRT^2_2 and COH, where COH is the so-called cohesive principle which states that for any sequence (R_i) of subsets of $\mathbb{N}$, there is a set which for each i is up to finite error a subset of either R_i or its complement. This work sparked a major investigation into the logical separation of SRT^2_2 from COH which was completed a few years ago by Monin and Patey [MP21], who showed that SRT^2_2 does not imply COH even in ω-models of RCA_0, the converse nonimplication having been established earlier in [HJKH$^+$08]. One is led to wonder what relationship $\mathsf{Ind}\,\mathscr{E}_2$ has to COH: clearly COH cannot imply $\mathsf{Ind}\,\mathscr{E}_2$ in ω-models, but whether COH is reducible in any sense to $\mathsf{Ind}\,\mathscr{E}_2$ is open, as is the question of the separation between $\mathsf{Ind}\,\mathscr{E}_k$ and SRT^2_k under $\leq_\mathrm{c}$. We conjecture that

Conjecture 6.2. $\mathsf{COH} \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ and $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{c}} \mathsf{SRT}^2_{\mathbb{N}}$, or at least $\mathsf{Ind}\,\mathscr{E}_k \not\leq_{\mathrm{c}} \mathsf{SRT}^2_k$.

Of course, if either statement is false, then the other would immediately follow. We briefly mention a way to view $\mathsf{Ind}\,\mathscr{E}_k$ in terms of RT^1_k which might suggest a route to the resolution of both conjectures. Several authors have observed that $\mathsf{COH} \equiv_{\mathrm{W}} (\widehat{\mathsf{RT}^1_2})^{\mathrm{FE}}$, where P^{FE} is the "finite error" version of P with $\operatorname{dom} P^{\mathrm{FE}} = \operatorname{dom} P$ and $x \in P^{\mathrm{FE}}(p)$ iff there is a $y \in P(p)$ with x and y differing only on a finite set. (See for instance [DM22, Corollary 8.4.15].) The next definition was inspired by the principle RCOH, a weakening of COH introduced in [CDHP20] which corresponds to "solving infinitely many columns" of an $(\widehat{\mathsf{RT}^1_2})^{\mathrm{FE}}$-instance.

Definition 6.3. The *weak parallelization* of a problem P is the problem $\widetilde{P}$ such that $\operatorname{dom} \widetilde{P} = \operatorname{dom} \widehat{P}$, and where the solutions to the instance $\langle p_0, p_1, \ldots \rangle$ are all sets of the form

$$\bigcup_{n \in A} \{ \langle n, x \rangle \, : \, x \in P(p_n) \},$$

where A is an infinite subset of $\mathbb{N}$.

In other words, $\widetilde{P}$ picks infinitely many parallel instances of P to solve out of a given instance of $\widehat{P}$. Thus $\mathsf{Ind}\,\mathscr{E}_k$ is an a priori stronger variant of $\widetilde{\mathsf{RT}^1_k}$ in which the solutions of the parallel RT^1_k-instances represented in an $\widetilde{\mathsf{RT}^1_k}$-solution are all of the same color. However, by nonuniformly picking a color shared by infinitely many columns of a solution of $\widetilde{\mathsf{RT}^1_k}$, it is not hard to see that $\mathsf{Ind}\,\mathscr{E}_k \equiv_{\mathrm{c}} \widetilde{\mathsf{RT}^1_k}$. Moreover, $\mathsf{SRT}^2_k \equiv_{\mathrm{c}} \widetilde{\mathsf{SRT}^1_k}$ by additionally computing a homogeneous set using a standard argument as in the proof of Lemma 5.3. Then Conjecture 6.2 can be rephrased in these terms as

Conjecture 6.4. $(\widehat{\mathsf{RT}^1_2})^{\mathrm{FE}} \not\leq_{\mathrm{c}} \widetilde{\mathsf{RT}^1_2}$ and $\widetilde{\mathsf{RT}^1_k} \not\leq_{\mathrm{c}} \widetilde{\mathsf{SRT}^1_k}$.

Finally, there are of course many indivisible structures other than $\mathbb{Q}$ and $\mathscr{E}$ which could be investigated along similar lines as in the present work, such as the Rado graph or nonscattered linear orders. As a followup to Proposition 3.6, it would be interesting to obtain a characterization of the linear orders $\mathcal{L}$ such that $\mathsf{Ind}\,\mathcal{L} \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$. It might also be fruitful to continue the study of general properties of indivisibility problems initiated in Section 3. Arguably one of the most basic questions we leave open is the following:

Question 6.5. Is it the case that $\mathsf{Ind}\,\mathcal{S}_{k+1} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_k$ for every indivisible structure $\mathcal{S}$?

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