Probabilistic automatic complexity

Kenneth Gill

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- **NFA complexity**, $A_N(x)$: least number of states of an NFA accepting x and having a unique accepting path of length $|x|$ (Hyde '13)

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- reads an input word sequentially starting in an initial state;
- changes states probabilistically. Formally given by stochastic matrices P_σ with $(P_\sigma)_{ij}$ = prob. of moving from state s_i to state s_j when reading letter σ;
- assigns an **acceptance probability** $\rho(x)$ to each word x, i.e., the prob. of ending in an accepting state after reading x.

PFA complexity

The **gap function** of the PFA M is

$$\mathrm{gap}_M(x) = \min\{\, \rho_M(x) - \rho_M(y) : |y| = |x| \text{ and } y \neq x \,\}.$$

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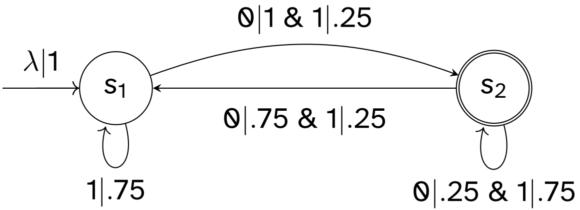
Example: the NFA complexity of $x = 0^6 1^8 0$ is 8, but $A_P(x) = 2$ via



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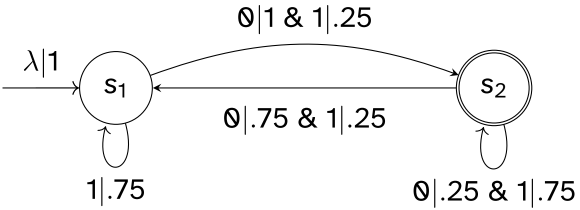

For this PFA, $\rho(x) \approx 0.625$ with $\mathrm{gap}(x) \approx 4 \times 10^{-6}$ (!).

PFA complexity with required gap

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$$A_P(x) = A_{P,0}(x) \leq A_{P,\delta}(x) \leq A_D(x) \qquad \text{for all } x, \delta,$$

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Not a tight bound! Only met by constant strings so far.
In fact $A_P(x) \le 3 \ \forall\, |x| \le 9$.

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A_P is not known to be computable—yet—but $A_{P,\delta}$ is (“almost everywhere”):

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Proof via computable analysis. (Thanks to Jake Canel for suggesting the approach.)

Classification of binary strings with $A_P = 2$

Theorem (G. '23)

For a binary string w, $A_P(w) = 2 \iff w$ is of the form

$$0^n 1^m, \qquad 0^n 1^m 0, \qquad 0^n (10)^m, \qquad \text{or} \quad 0^n 1 (01)^m$$

for some $n, m \geq 0$, or is the bit-flip of one of the above.

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The proof of this theorem shows that a generic 2-state PFA describes an infinite family of strings of similar structure.

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on $[0, 1]$ and a number x_0 such that for any w,

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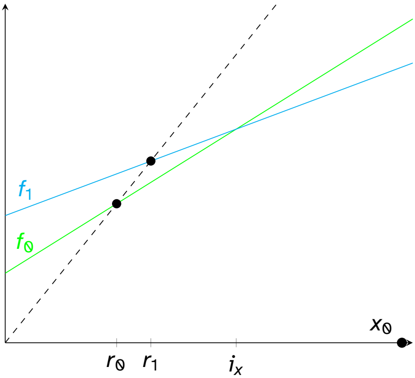
Idea of forward direction: For each n, find the sequence of n compositions of f_0 and f_1 attaining the highest possible value.

Illustration of forward direction: Positive slopes



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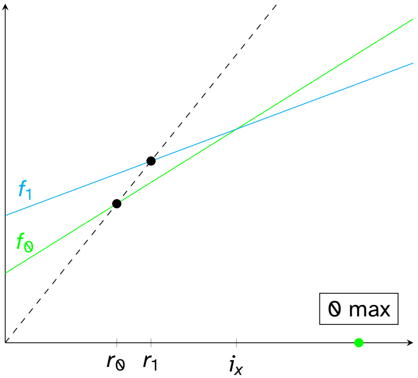

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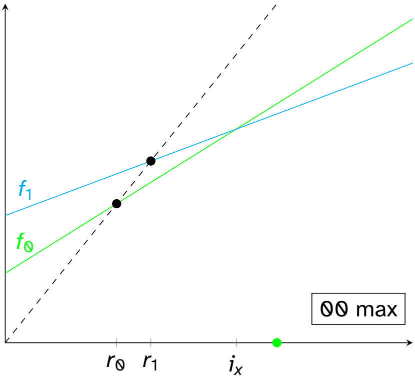

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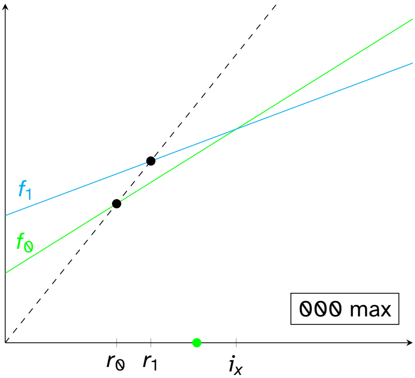

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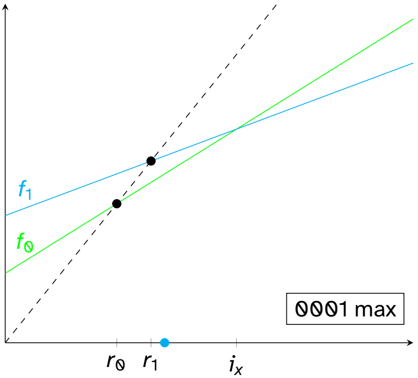

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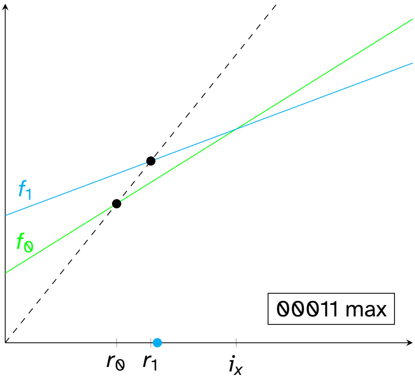

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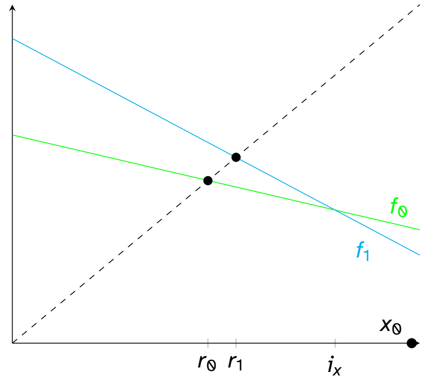

Negative slopes: $1^n(01)^m$, $0^n1(01)^m$ $\forall m$

Max-min-max-min pattern

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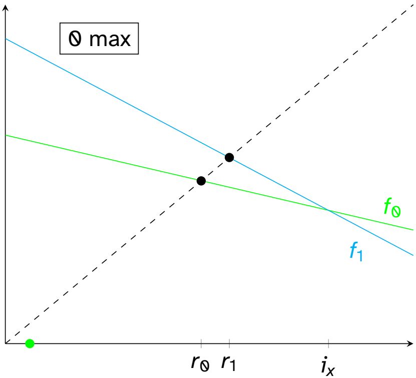

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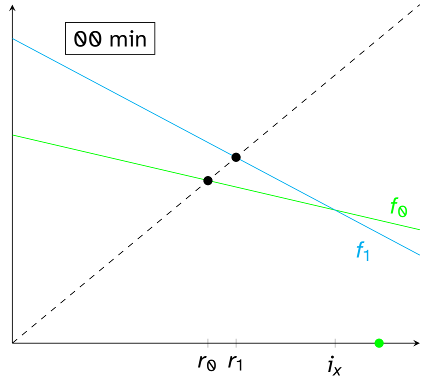

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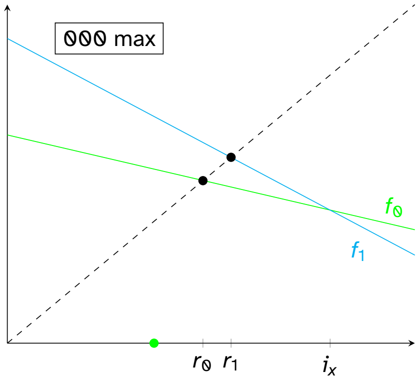

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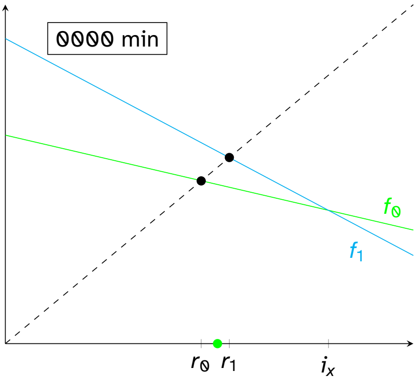

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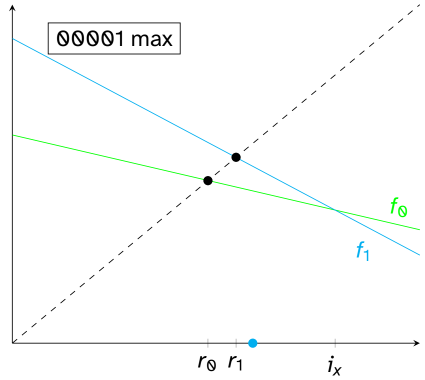

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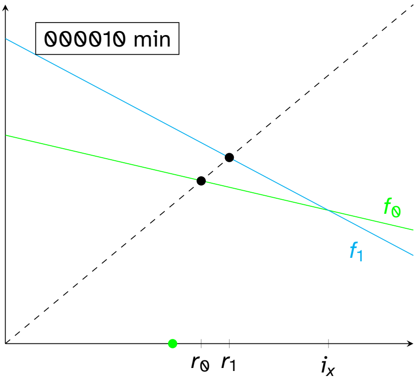

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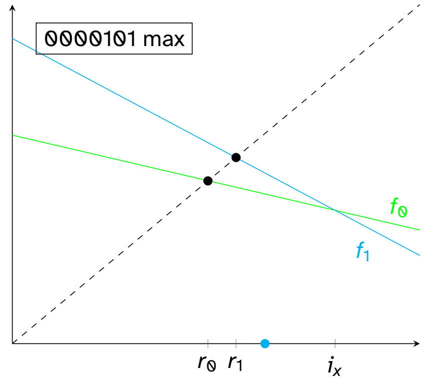

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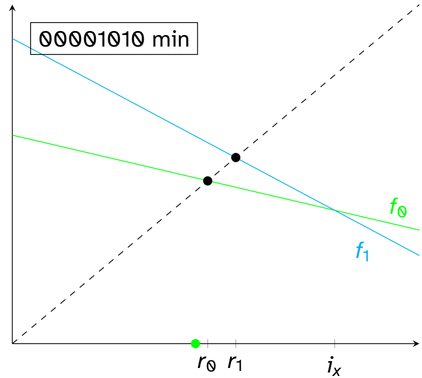

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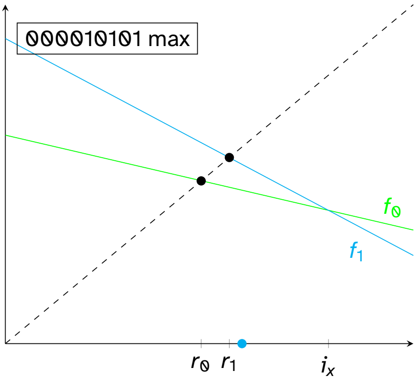

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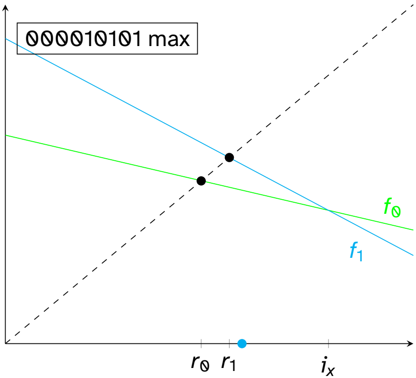

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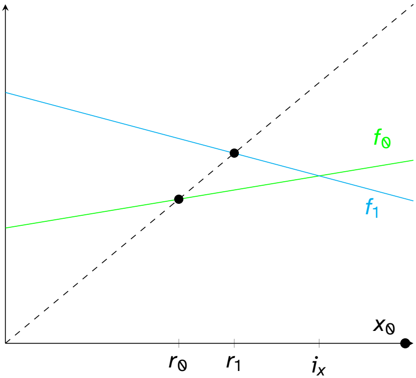

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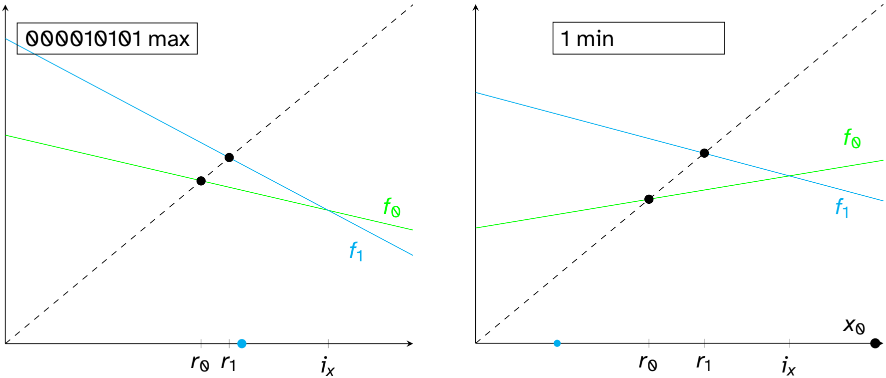

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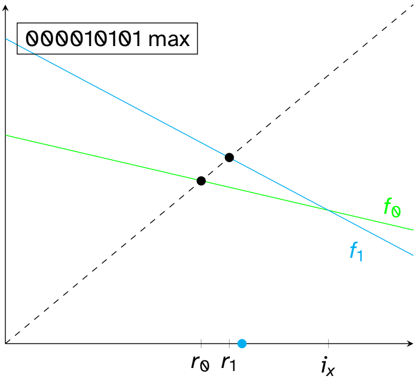

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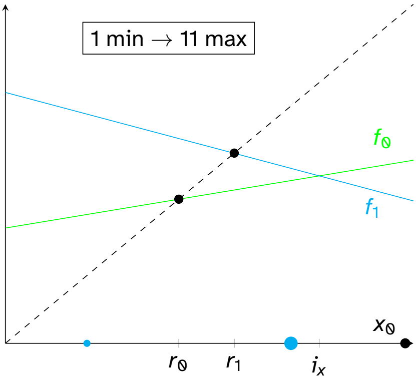

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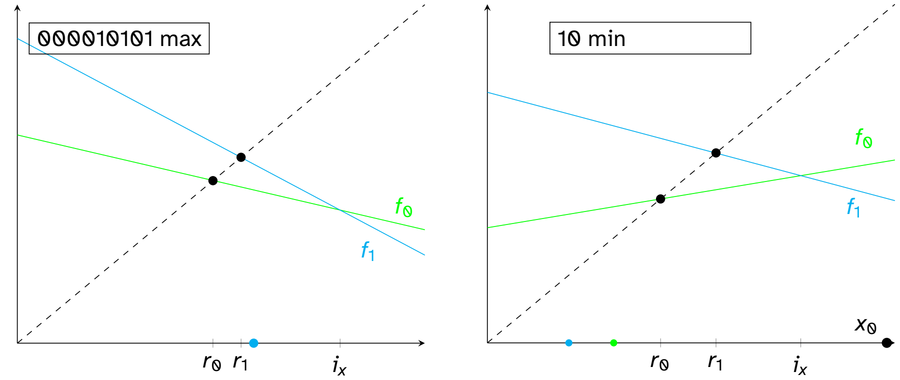

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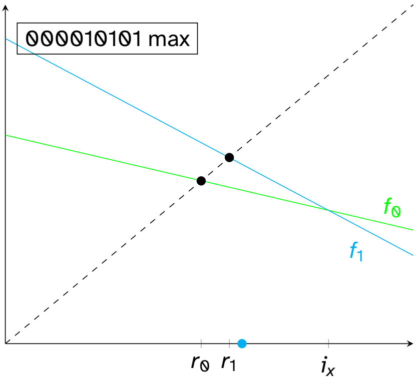

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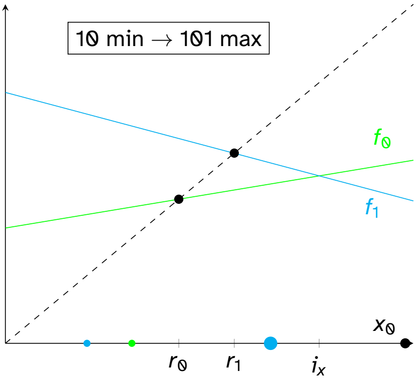

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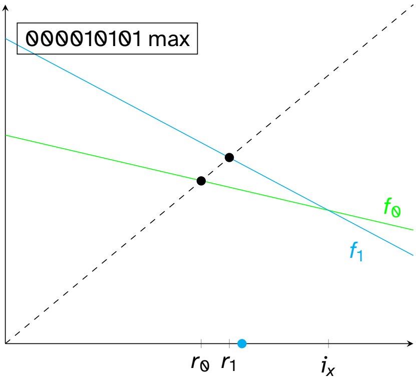

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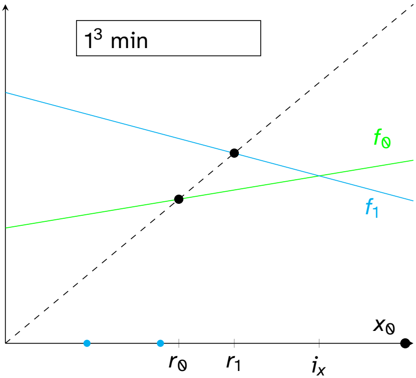
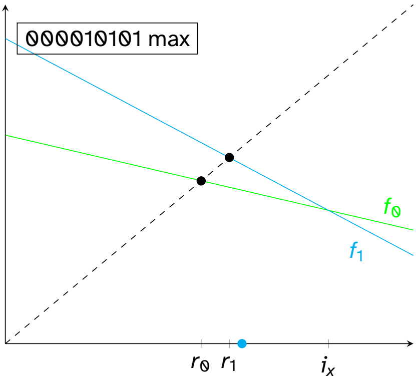


Mixed slopes: 1^n0^m1 $\forall m$

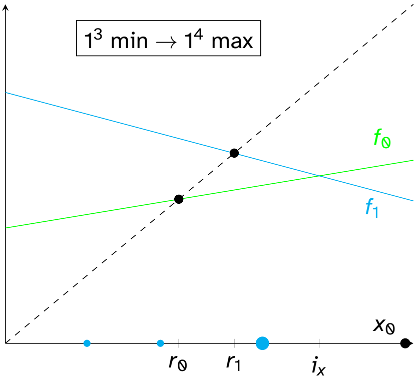
Stay min as long as possible, then apply f_1

Other subcases of the forward direction



Negative slopes: $1^n(01)^m$, $0^n 1(01)^m$ $\forall m$

Max-min-max-min pattern



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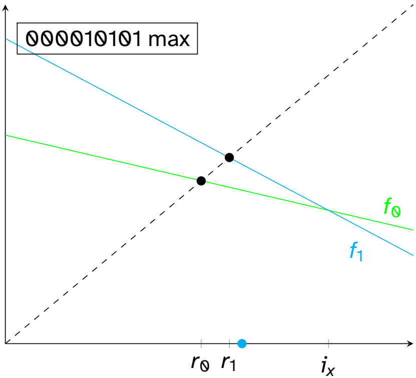

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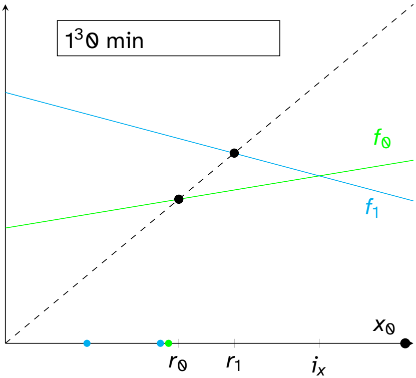

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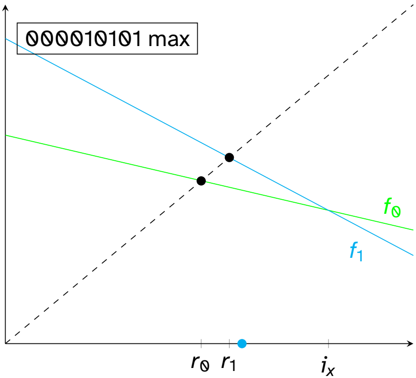

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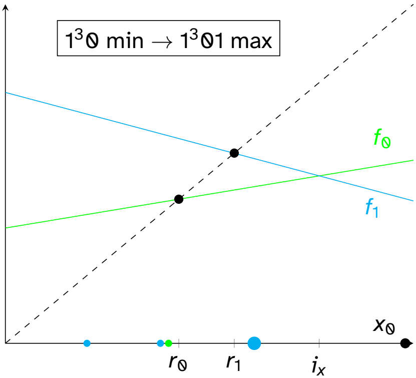

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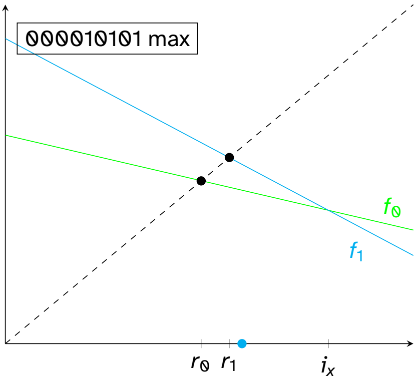

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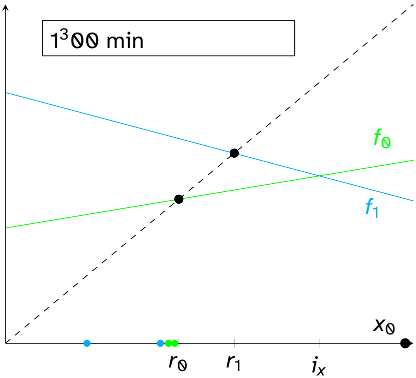

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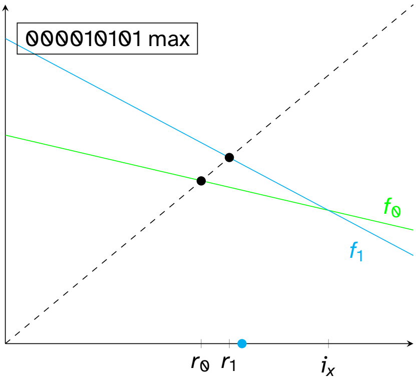

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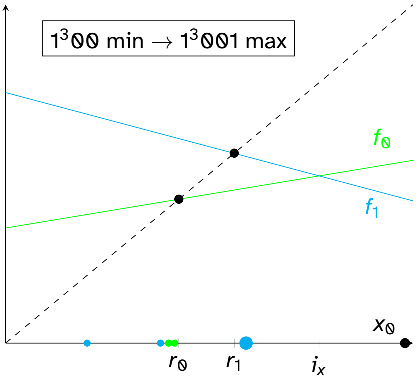

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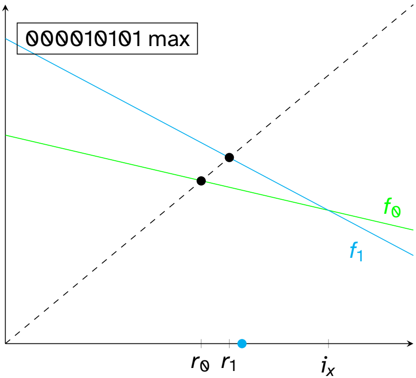

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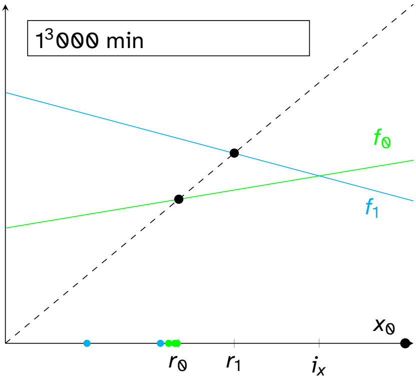

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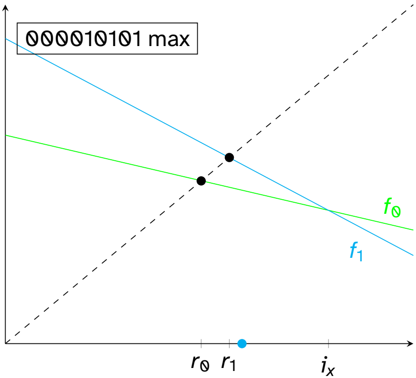

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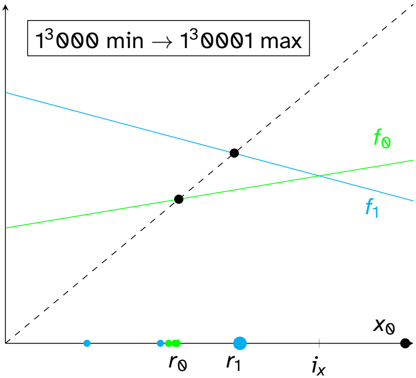

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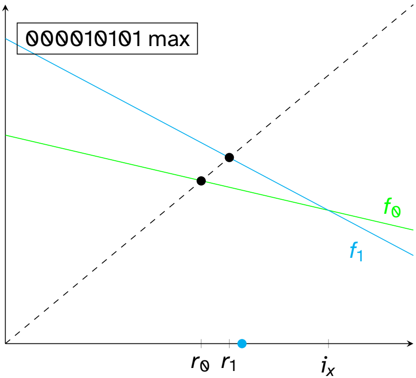

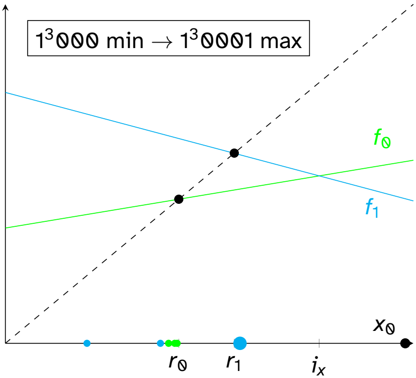

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Stay min as long as possible, then apply f_1

(Proof: *really* long!)

Further directions

- A_P computable? (likely yes: details pending)
- A_P unbounded? If so, tight asymptotic bound? Same questions for $A_{P,\delta}$.
- Use the max-gap function $\gamma^k(w)$ as a computable parametrized complexity measure instead?

References

- K. Gill, Probabilistic automatic complexity of finite strings, submitted (2024).
- K. Gill, *Two studies in complexity*, Ph.D. dissertation, Penn State University, 2023.
- K. Hyde, *Nondeterministic finite state complexity*, MA thesis, University of Hawai'i, Manoa, 2013.
- J. Shallit and M.-w. Wang, Automatic complexity of strings, *J. Autom. Lang. Comb.* **6**(4) (2001), 537–554.

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